Azumaya Algebras

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1 Introduction: Central Simple Algebras

Azumaya algebras are introduced as generalized or global versions of central simple algebras. So the first part of this seminar will be about central simple algebras.

Definition 1.1. A ring R is called simple if 0 and R are the only two-sided ideals.

Simple rings are only interesting if they are noncommutative because we have the following:

Proposition 1.2. *If R is a commutative, simple ring. Then R is a field.*

Proof. Take x a nonzero element in R, then Rx is a nonzero twosided ideal and hence is equal to R. In particular $1 \in Rx$ and thus x is invertible. $\square$

Definition 1.3. Let k be a field and A a finite dimensional associative k-algebra. Then A is called a central simple algebra (CSA) over k if A is a simple ring and $Z(A) = k$

Note that the inclusion of k in the center of A is automatic as A is a k algebra.

Example 1.4. Let n be some natural number, then the matrix ring $M_n(k)$ is a CSA over k. It obviously has dimension n^2 over k so we only need to check that it is central and simple.
To see this, let e_{ij} denote the matrix with a 1 at position (i, j) and zeroes at all other positions, i.e.

$$e_{ij} = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ 0 & \ddots & & & \\ \vdots & & 1 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix}$$

Then for a matrix $e_{ii}M = Me_{ii}$ for all i implies that M is diagonal and $e_{ij}M = Me_{ij}$ for all i and j implies that all entries on the diagonal must be the same.

Hence a central matrix must be a scalar matrix and obviously all scalar matrices are central. In a similar way one can show that any nonzero ideal must be $M_n(k)$ because suppose I is some nonzero ideal and $M \in I \setminus \{0\}$. Suppose $m_{ij} \neq 0$ then $e_{ii} = (m_{ij})^{-1} \cdot e_{ii} M e_{ij} \in I$ and similarly for all l: $e_{ll} = e_{li} e_{ii} e_{il} \in I$, hence

$$\mathrm{Id}_n = \sum_{l=1}^{n} e_{ll} \in I$$

Although not every central simple algebra over a field is a matrix ring over this field, the next theorems show that they are closely related to matrix rings.

Theorem 1.5 (Wedderburn (it is a special case of the more general Artin-Wedderburn Theorem))**.** *Let A be a CSA over k. Then there is a unique division algebra D (i.e. a division ring which is a algebra over k) and a positive integer n such that*

$$A \cong M_n(D)$$

Remark. The division algebra D in the above theorem is automatically a central k-algebra because

$$k = Z(A) = Z(M_n(D)) = Z(D)$$

Corollary 1.6. *If k is algebraically closed then any CSA over k is a matrix ring.*

Proof. By the Wedderburn Theorem it suffices to prove that a finite dimensional division algebra D over k is automatically trivial (i.e. $D = k$). So suppose by way of contradiction that $x \in D \setminus k$. As x is invertible in D there is an inclusion $k(x) \subset D$. As D is finite dimensional over k, so is $k(x)$ and thus $k(x) = k[x]$ is a finite algebraic extension of k. A contradiction with the fact that k is algebraically closed. $\square$

Corollary 1.7. *The dimension of a CSA over a field is always a square.*

Proof. If A is a CSA over k, then obviously $A \otimes_k \overline{k}$ is a CSA of the same dimension over the algebraic closure $\overline{k}$. But by the above the latter must be a matrix ring. $\square$

Another useful notion is that of a splitting field:

Theorem 1.8. *Let A be a CSA over k of dimension n^2. A splitting field for A is a field extension F of k such that $A \otimes_k F \cong M_n(F)$. Such a splitting field always exists and can be chosen to be separable over k, in particular we can choose $F = k$ if k is separably closed.*

Proof. By the Wedderburn Theorem $A \cong M_i(D)$ for some division algebra D over k. As $M_i(M_j(F)) \cong M_{i \cdot j}(F)$ and $M_i(D) \otimes_k F \cong M_i(D \otimes_k F)$, it hence suffices to prove that any (central) division algebra D over k admits a splitting field. It is then known that F can be chosen as any maximal subfield of D in which case $[F : k] = n$. Moreover at least one of these maximal subfields is separable over k. See [Coh03, Corollary 5.1.12, Corollary 5.2.7, Theorem 5.2.8] for the details. $\square$

The following result is obvious but interesting:

Proposition 1.9. *If A and B are CSAs over k, then so is $A \otimes_k B$.*

Lastly we say something about 4-dimensional CSAs. We have the following theorem

Theorem 1.10. *Let k be a field of characteristic different from 2 and let A be a 4-dimensional k-algebra, then the following are equivalent*

i) A is a CSA over k

ii) There are $a, b \in k \setminus \{0\}$ and a k-basis $\{1, i, j, k\}$ for A such that the multiplication on A is given by

- $i^2 = a$
- $j^2 = b$
- $ij = k = -ji$

Proof. We quickly sketch both directions

$ii) \Rightarrow i)$ This is done by some explicit computations similar to the example of a matrix ring.

$i) \Rightarrow ii)$ By the Wedderburn Theorem, A is either a matrix ring or a division algebra. In the first case we choose

$$i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, a = -1, b = 1$$

for the second case we first note that for any $x \in A$: $1,x$ and x^2 are necessarily linearly dependent. Next we construct a basis $\{1, i', j', i'j'\}$ with $(i')^2 = a', (j')^2 = b'$ but where the last condition might fail. As a last step we can tweek this last basis in order for $ij = -ji$ to hold.

$\square$

Remark. It is known that in case $k = \mathbb{R}$ every CSA of dimension 4 is isomorphic to either $M_2(\mathbb{R})$ (with $a = -1$, $b = 1$) or $\mathbb{H}$, the Hamilton quaternions (with $a = b = -1$). By Theorem 1.8 the latter must have a splitting field of dimension 2 over $\mathbb{R}$ and indeed $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$.

2 Azumaya algebras over local rings

We now generalize CSAs over a field to Azumaya algebras over local rings. There are several equivalent ways to define Azumaya algebras. Following the book [Mil80] we start with the following rather technical definition:

Definition 2.1. Let R be a commutative local ring (this will be the case throughout this section) and let A be an associative R-algebra such that $R \to A$ identifies R with a subring of $Z(A)$ (i.e. the structure morphism is injective). Then A is called an Azumaya algebra over R is A is free of finite rank l as an R-module and if the following map is an isomorphism:

$$\phi_A : A \otimes_R A^{op} \to End_R(A) : a \otimes a' \mapsto (x \mapsto axa')$$

where A^{op} is the opposite algebra to A (i.e. the same additive structure and the multiplicative structure given by by $a \bullet b := b \cdot a$).

Remark.

- As we require A to be free over R, the inclusion $R \subset Z(A)$ is automatic.
- ϕ_A always is an R-algebra morphism, so only the bijectivity in the definition is a nontrivial condition.

In the case where $R = k$ is a field we have the following:

Proposition 2.2. *If A is a CSA over a field k, then A is Azumaya over k.*

Proof. Let $\dim_k(A) = l$ then ϕ_A is a morphism between k-algebras which both have dimension l^2 over k. Hence it suffices to check injectivity. Note that $A \otimes_k A^{op}$ is a CSA over k hence $\ker(\phi_A) = 0$ or $A \otimes_k A^{op}$. As the second option is obviously false we have proven injectivity of ϕ_A. $\square$

The other direction is also true and follows from the following proposition

Proposition 2.3. *Let A be an Azumaya algebra over R, then $Z(A) = R$ and there is a bijection between the (two-sided) ideals of A and the ideals of R:*

$$\begin{array}{rcl} \{ \text{ Ideals of } A\} & \overset{1-1}{\longleftrightarrow} & \{ \text{ Ideals of } R\} \\ \mathcal{I} & \mapsto & \mathcal{I} \cap R \\ \mathcal{J}A & \leftarrow\!\shortmid & \mathcal{J} \end{array}$$

Proof. Let $\psi \in End_R(A)$ and $c \in Z(A)$ then for all $a \in A$ we have $c\psi(a) = \psi(ca) = \psi(ac) = \psi(a)c$ because ψ is given by multiplication by elements in A as A is Azumaya. Similarly $\psi(\mathcal{I}) \subset \mathcal{I}$ for each ideal $\mathcal{I}$ of A. Now let $1 = a_1, \dots, a_l$ be a basis for A as an R-module and define $\chi_i \in End_R(A)$ by $\chi_i(a_j) = \delta_{ij}$. Write $c = \sum_i r_i a_i$ with all $r_i \in R$, then

$$c = 1 \cdot c = \chi_1(a_1)c = \chi_1(a_1 c) = \chi_1(1 \cdot c) = \chi_1\left(\sum_{i=1}^{l} r_i a_i\right) = r_1 \in R$$

Now we check the bijection between the sets of ideals. As the maps are well defined, it suffices to prove $\mathcal{I} = (\mathcal{I} \cap R)A$ and $\mathcal{J} = \mathcal{J}A \cup R$. Both equalities are trivial to check. $\square$

Corollary 2.4. *An Azumaya algebra over a field is a CSA.*

Proposition 2.5. *Let (R, m), (R', m') be commutative local rings and let A be a free R-module of rank l. Assume there is a morphism $R \to R'$ then:*

i) If A is Azumaya over R then $A \otimes_R R'$ is Azumaya over R'.

ii) If $A \otimes R/m$ is Azumaya (hence CSA) over R/m then A is Azumaya over R.

Proof. We have the following commutative diagram:

$$\begin{tikzcd} \phi_A \otimes R' : (A \otimes_R A^{op}) \otimes_R R' \arrow[r] \arrow[d, "\cong"'] & End_R(A) \otimes_R R' \arrow[d, "\cong"'] \\ \phi_{A \otimes R'} : (A \otimes_R R') \otimes_{R'} (A \otimes_R R')^{op} \arrow[r] & End_{R'}(A \otimes_R R') \end{tikzcd}$$

The first statement follows immediately from this diagram. For the second statement note that surjectivity of $\phi_A \otimes_R R/m$ implies surjectivity of ϕ_A by Nakayama's Lemma. For the injectivity we need a technical Lemma, e.g. [Mil80, Lemma IV.1.11] $\square$

Corollary 2.6. *Let A be a free module of rank l over (R, m) and let $k = R/m$, then the following are equivalent:*

- *A is Azumaya over R*
- *$A \otimes k$ is a CSA over k*
- *$A \otimes \overline{k} \cong M_n(\overline{k})$*

In particular $l = n^2$ for some $n \in \mathbb{N}$

Corollary 2.7.
- *The tensor product of two Azumaya algebras is an Azumaya algebra.*
- *$M_n(R)$ is Azumaya over R*

We now state the main result of this section

Proposition 2.8 (Skolem-Noether)**.** *Let A be Azumaya over R, then every $\psi \in Aut_R(A)$ is inner. I.e. for any such ψ there is a unit $u \in A^*$ such that $\psi(a) = uau^{-1}$.*

Proof. Given $\psi \in Aut_R(A)$, there are two different ways to turn A into an $A \otimes_R A^{op}$-module:

$$\begin{cases} (a_1 \otimes a_2)a &= a_1 a a_2 \\ (a_1 \otimes a_2)a &= \psi(a_1) a a_2 \end{cases}$$

Denote the resulting $A \otimes_R A^{op}$-modules by A, respectively A'. Both $\overline{A'} := A' \otimes_R R/m$ and $\overline{A}$ are simple $\overline{A} \otimes_{\overline{R}} \overline{A}^{op}$-modules. This is based on the fact that $\overline{A} \otimes_{\overline{R}} \overline{A}^{op}$-submodules of $\overline{A}$ or $\overline{A'}$ correspond to two-sided ideals of the central simple algebra $\overline{A}$ (the argument for $\overline{A'}$ uses the fact that ψ is not just an endomorphism but an automorphism).
By Proposition 1.9: $\overline{A} \otimes_{\overline{R}} \overline{A}^{op}$ is a CSA over $\overline{R} = R/m$ and thus it is of the form $M_n(D)$ for some division algebra D over $\overline{R}$. All simple modules over $M_n(D)$ are of the form D^n, so there must be an isomorphism of $\overline{A} \otimes_{\overline{R}} \overline{A}^{op}$-modules:

$$\overline{\chi} : \overline{A} \to \overline{A'}$$

We now claim that this lifts to a surjective $A \otimes_R A^{op}$-module morphism $\chi : A \to A'$.

First suppose the claim holds, then setting $u = \psi(1)$ gives:

$$\psi(a)u = (a \otimes 1)u = \chi((a \otimes 1)1) = \chi(a) = \chi((1 \otimes a)1) = (1 \otimes a)\chi(1) = ua$$

Surjectivity of χ gives the existence of an $a_0 \in A$ such that $\chi(a_0) = 1$, hence

$$1 = \chi(a_0) = \chi((1 \otimes a_0)1) = ua_0$$

implying that u is invertible in A.

Now we prove the claim: Note that we have the following diagram of $A \otimes_R A^{op}$-module morphisms:

$$\begin{tikzcd}
 & A \arrow[d] \\
 & \overline{A} \arrow[d, "\overline{\chi}"] \\
A' \arrow[r, two heads] & \overline{A'}
\end{tikzcd}$$

so the existence of χ follows if we can prove that A is a projective $A \otimes_R A'$-module. As A is free as an R-module there is an R-module morphism $g : A \to R$ such that $g(r) = r$. As A is Azumaya we have $A \otimes_R A^{op} \cong End_R(A)$ and A is a direct summand of $End_R(A)$ via

$$A \xrightarrow{a \mapsto (a' \mapsto g(a')a)} End_R(A) \xrightarrow{f \mapsto (f(1))} A$$

Finally surjectivity of χ follows from Nakayama's Lemma. $\square$

3 Azumaya algebras over schemes

Throughout this section, let X be a locally Noetherian scheme

Proposition 3.1. *Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra of finite type as an $\mathcal{O}_X$-module. Then the following are equivalent:*

i) *$\mathcal{A}$ is a locally free $\mathcal{O}_X$ module and $\phi_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A}^{op} \to End_{\mathcal{O}_X}(\mathcal{A})$ is an isomorphism*

ii) *$\mathcal{A}_x$ is Azumaya over $\mathcal{O}_{X,x}$ for each point x in X*

iii) *$\mathcal{A}_x$ is Azumaya over $\mathcal{O}_{X,x}$ for each closed point x in X*

iv) *$\mathcal{A}$ is a locally free $\mathcal{O}_X$ module and $\mathcal{A}_x \otimes \mathcal{O}_{X,x}/m_x$ is a CSA over $k(x) = \mathcal{O}_{X,x}/m_x$*

v) *There is a covering $(U_i \to X)$ for the étale topology such that for each $i : \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong M_{r_i}(\mathcal{O}_{U_i})$ for some r_i*

vi) *There is a covering $(U_i \to X)$ for the fppf (flat and of finite presentation) topology such that for each $i : \mathcal{A} \otimes \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong M_{r_i}(\mathcal{O}_{U_i})$ for some r_i*

Remark. Note that in the above proposition we made some abuse of notation: if $(U_i \xrightarrow{f_i} X)$ is a covering in the étale or flat topology, then $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i}$ is short hand notation for $f_i^* \mathcal{A}$

Proof. We prove $i) \Leftrightarrow ii) \Leftrightarrow iii)$ and $ii) \Rightarrow v) \Rightarrow vi) \Rightarrow iv) \Rightarrow ii)$.

$i) \Leftrightarrow ii)$ As $\mathcal{A}$ is locally free we have $(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{op})_x = \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{A}_x)^{op}$ and $End_{\mathcal{O}_X}(\mathcal{A})_x = End_{\mathcal{O}_{X,x}}(\mathcal{A}_x)$. And $\phi_{\mathcal{A}}$ is an isomorphism if and only if all $(\phi_{\mathcal{A}})_x = \phi_{\mathcal{A}_x}$ are isomorphisms.

$ii) \Rightarrow iii)$ obvious

$iii) \Rightarrow ii)$ If η is some non-closed point, then there is a closed point $x \in \overline{\eta}$ (on a locally Noetherian scheme we have existence of closed points: [Sta15, Tag 01OU, Lemma 27.5.9.]). Then there is a (non-local) morphism of local rings $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\eta}$. The result then follows from Proposition 2.5.

$ii) \Rightarrow v)$ Let x be a point in X, let $k(x) = O_{X,x}/m_x$ with separable closure $\widetilde{k(x)}$ and let $\overline{x} : \mathrm{Spec}(\widetilde{k(x)}) \to X$ be the associated geometric point. Recall from the previous lecture that the étale stalk $\mathcal{O}_{X,\overline{x}}$ is a module over the Zariski stalk $\mathcal{O}_{X,x}$, because it is it's strict Henselization. In particular we can apply Proposition 2.5 to obtain that $\mathcal{A}_x \otimes \mathcal{O}_{X,\overline{x}}$ is Azumaya over the strict Henselian (local) ring $\mathcal{O}_{X,\overline{x}}$. The residue field of $\mathcal{O}_{X,\overline{x}}$ is given by $\widetilde{k(x)}$ and hence is separably closed. By Theorem 1.8, this implies that $\mathcal{A}_x \otimes \widetilde{k(x)}$ is split. [Mil80, Proposition IV.1.6] then implies that $\mathcal{A}_x \otimes \mathcal{O}_{X,\overline{x}}$ is itself split, i.e. $\mathcal{A}_x \otimes \mathcal{O}_{X,\overline{x}} \cong M_r(\mathcal{O}_{X,\overline{x}})$. From this it follows that there is an étale morphism $U \xrightarrow{f} X$ whose image contains x such that $f^*\mathcal{A} \cong M_r(U)$.

(Details: as $\mathcal{A}_x \otimes \mathcal{O}_{X,\overline{x}} \cong M_r(\mathcal{O}_{X,\overline{x}})$ there is some basis $(e_{ij})_{i,j=1,\ldots,r}$ (suggestive notation!) for $\mathcal{A}_x \otimes \mathcal{O}_{X,\overline{x}}$ as an $\mathcal{O}_{X,\overline{x}}$ module such that the multiplicative relations are given by $e_{ij}e_{jl} = e_{il}$. As $\mathcal{A}$ is locally free (in the Zariski and hence also étale topology) we have $\mathcal{A}_x \otimes \mathcal{O}_{X,\overline{x}} \cong \mathcal{A}_{\overline{x}}$ and there must exist some étale environment V of x such that there is a basis $(c_{ij})_{i,j=1,\ldots,r}$ for $\mathcal{A}(V)$ for which $(c_{ij})_{\overline{x}} = e_{ij}$. (The fact that the c_{ij} give a basis for $\mathcal{A}(V)$ is heavily based on the locally freeness of $\mathcal{A}$.)
For each i, j, l the relation

$$(c_{ij})_x \cdot (c_{jl})_x = (c_{il})_x$$

implies that there is some étale environment $V_{ijl} \to V \to X$ of x for which

$$c_{ij}|_{V_{ijl}}\ c_{jl}|_{V_{ijl}} = c_{il}|_{V_{ijl}}$$

we can then set $U = V_{000} \times_V V_{001} \times_V \ldots \times_V V_{rrr}$.

$v) \Rightarrow vi)$ Trivial as each étale morphism is flat and of finite presentation.

$vi) \Rightarrow iv)$ Let $(U_i \stackrel{f_i}{\to} X)$ be a covering for the flat topology such that for each $i : \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong M_{r_i}(\mathcal{O}_{U_i})$. We first check that $\mathcal{A}$ is locally free. Let $U = \amalg U_i$, then $\amalg f_i : U \to X$ is surjective and flat (hence by definition faithfully flat). In particular the fact that $f^*\mathcal{A}$ is a flat $\mathcal{O}_U$-module (this is true because it is finitely generated and locally free) implies that $\mathcal{A}$ is a flat and hence locally free $\mathcal{O}_X$-module.

(Details: suppose we are given a short exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

As f is flat the following is exact as well:

$$0 \to f^*\mathcal{F} \to f^*\mathcal{G} \to f^*\mathcal{H} \to 0$$

Now $- \otimes_{\mathcal{O}_U} f^*\mathcal{A}$ is an exact functor pullback commutes with tensor product, so we get the exact sequence

$$0 \to f^*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}) \to f^*(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{A}) \to f^*(\mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{A}) \to 0$$

and as f is <u>faithfully</u> flat this implies that

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{A} \to 0$$

is itself exact and hence $\mathcal{A}$ is a flat $\mathcal{O}_X$-module.)

Next let $x \in X$ be chosen at random. We check that $\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is a CSA over $k(x)$. Take i such that x is in the image of f_i, say $f_i(y) = x$

for some $y \in U_i$. Then $(f_i^*\mathcal{A})_y = \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{U_i,y}$ and thus

$$\begin{aligned}
\left(\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} k(x)\right) \otimes_{k(x)} k(y) &\cong \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \left(k(x) \otimes_{k(x)} k(y)\right) \\
&\cong \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \left(\mathcal{O}_{U_i,y} \otimes_{\mathcal{O}_{U_i,y}} k(y)\right) \\
&\cong \left(\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{U_i,y}\right) \otimes_{\mathcal{O}_{U_i,y}} k(y) \\
&\cong (f_i^*\mathcal{A})_y \otimes_{\mathcal{O}_{U_i,y}} k(y) \\
&\cong M_{r_i}(k(y))
\end{aligned}$$

This implies that some base change of $\mathcal{A}_x \otimes k(x)$ is a CSA and hence $\mathcal{A}_x \otimes k(x)$ is itself a CSA.

iv) $\Rightarrow$ *ii)* This was already proved in Proposition 2.5.

□

We end this seminar by the following version of the Skolem-Noether theorem:

Proposition 3.2. *Let $\mathcal{A}$ be Azumaya on X, then every $\psi \in Aut(\mathcal{A})$ is locally, for the Zariski topology on X an inner automorphism. I.e. there is a Zariski-open covering $(U_i \to X)$ such that $\psi|_{U_i}$ is of the form $a \mapsto uau^{-1}$ for some $u \in \Gamma(U_i, \mathcal{A})^*$*

Proof. Let $x \in X$, then by the Skolem-Noether Theorem for local rings (Proposition 2.8) there is a $u_x \in \mathcal{A}_x^*$ such that $\psi_x(a_x) = u_x^{-1} a_x u_x$ for all $a_x \in \mathcal{A}_x$. As u_x is invertible in $\mathcal{A}_x$, there is some open environment U of x such that u_x is given by the fibre of some invertible $u \in \Gamma(U, \mathcal{A})^*$. (by definition u_x being invertible asks for $u_x v_x = v_x u_x = 1_x$. Again by definition this implies that there are section u, v on some environment U of x such that $uv = vu = 1_U$ and $(u)_x = u_x$, $(v)_x = v_x$.) As $\mathcal{A}$ is locally free, we may assume $\mathcal{A}$ is free on U (shrinking U if necessary). I.e. there exist $a_1, \dots, a_{n^2} \in \Gamma(U, \mathcal{A})$ which give a basis for $\mathcal{A}|_U$ as a free $\mathcal{O}_U$-module. We now have two morphisms of $\mathcal{O}_U$-modules

$$\begin{aligned}
\psi|_U :&\mathcal{A}_U \to \mathcal{A}_U \\
\varphi :&\mathcal{A}_U \to \mathcal{A}_U : a \mapsto u^{-1}au
\end{aligned}$$

whose stalks agree at x. I.e. $\left(\psi|_U (a_i)\right)_x = (\varphi(a_i))_x$ for all $i = 1, \dots, n^2$. By again shrinking U if necessary we can assume $\psi|_U (a_i) = \varphi(a_i)$ for all i and hence $\psi|_U$ is inner. □

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