1 Introduction: Central Simple Algebras

Azumaya algebras are introduced as generalized or global versions of central simple algebras. So the first part of this seminar will be about central simple algebras.

**Definition 1.1.** A ring $R$ is called simple if 0 and $R$ are the only two-sided ideals.

Simple rings are only interesting if they are noncommutative because we have the following:

**Proposition 1.2.** If $R$ is a commutative, simple ring. Then $R$ is a field.

*Proof.* Take $x$ a nonzero element in $R$, then $Rx$ is a nonzero twosided ideal and hence is equal to $R$. In particular $1 \in Rx$ and thus $x$ is invertible. \qed

**Definition 1.3.** Let $k$ be a field and $A$ a finite dimensional associative $k$-algebra.

Then $A$ is called a central simple algebra (CSA) over $k$ if $A$ is a simple ring and $Z(A) = k$

Note that the inclusion of $k$ in the center of $A$ is automatic as $A$ is a $k$ algebra.

**Example 1.4.** Let $n$ be some natural number, then the matrix ring $M_n(k)$ is a CSA over $k$. It obviously has dimension $n^2$ over $k$ so we only need to check that it is central and simple.

To see this, let $e_{ij}$ denote the matrix with a 1 at position $(i, j)$ and zeroes at all other positions, i.e.

$$
e_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots \\ \vdots & 1 & \ddots \\ 0 & \cdots & 0 \end{bmatrix}$$

Then for a matrix $e_{ii}M = Me_{ii}$ for all $i$ implies that $M$ is diagonal and $e_{ij}M = Me_{ij}$ for all $i$ and $j$ implies that all entries on the diagonal must be the same.
Hence a central matrix must be a scalar matrix and obviously all scalar matrices are central. In a similar way one can show that any nonzero ideal must be $M_n(k)$ because suppose $I$ is some nonzero ideal and $M \in I \setminus \{0\}$. Suppose $m_{ij} \neq 0$ then $e_{ii} = (m_{ij})^{-1} \cdot e_{ii} M e_{ij} \in I$ and similarly for all $l$: $e_{ll} = e_{li} e_{ii} e_{il} \in I$, hence

$$\text{Id}_n = \sum_{l=1}^n e_{ll} \in I$$

Although not every central simple algebra over a field is a matrix ring over this field, the next theorems show that they are closely related to matrix rings.

**Theorem 1.5** (Wedderburn (it is a special case of the more general Artin-Wedderburn Theorem)). Let $A$ be a CSA over $k$. Then there is a unique division algebra $D$ (i.e. a division ring which is an algebra over $k$) and a positive integer $n$ such that

$$A \cong M_n(D)$$

**Remark.** The division algebra $D$ in the above theorem is automatically a central $k$-algebra because

$$k = Z(A) = Z(M_n(D)) = Z(D)$$

**Corollary 1.6.** If $k$ is algebraically closed then any CSA over $k$ is a matrix ring.

**Proof.** By the Wedderburn Theorem it suffices to prove that a finite dimensional division algebra $D$ over $k$ is automatically trivial (i.e. $D = k$). So suppose by way of contradiction that $x \in D \setminus k$. As $x$ is invertible in $D$ there is an inclusion $k(x) \subset D$. As $D$ is finite dimensional over $k$, so is $k(x)$ and thus $k(x) = k[x]$ is a finite algebraic extension of $k$. A contradiction with the fact that $k$ is algebraically closed.

**Corollary 1.7.** The dimension of a CSA over a field is always a square.

**Proof.** If $A$ is a CSA over $k$, then obviously $A \otimes_k \overline{k}$ is a CSA of the same dimension over the algebraic closure $\overline{k}$. But by the above the latter must be a matrix ring.

Another useful notion is that of a splitting field:

**Theorem 1.8.** Let $A$ be a CSA over $k$ of dimension $n^2$. A splitting field for $A$ is a field extension $F$ of $k$ such that $A \otimes_k F \cong M_n(F)$. Such a splitting field always exists and can be chosen to be separable over $k$, in particular we can choose $F = k$ if $k$ is separably closed.

**Proof.** By the Wedderburn Theorem $A \cong M_i(D)$ for some division algebra $D$ over $k$. As $M_i(M_j(F)) \cong M_{i \cdot j}(F)$ and $M_i(D) \otimes_k F \cong M_i(D \otimes_k F)$, it hence suffices to prove that any (central) division algebra $D$ over $k$ admits a splitting field. It is then known that $F$ can be chosen as any maximal subfield of $D$ in
which case \([F : k] = n\). Moreover at least one of these maximal subfields is separable over \(k\). See \cite{Coh03} Corollary 5.1.12, Corollary 5.2.7, Theorem 5.2.8 for the details.

The following result is obvious but interesting:

**Proposition 1.9.** If \(A\) and \(B\) are CSAs over \(k\), then so is \(A \otimes_k B\).

Lastly we say something about 4-dimensional CSAs. We have the following theorem

**Theorem 1.10.** Let \(k\) be a field of characteristic different from 2 and let \(A\) be a 4-dimensional \(k\)-algebra, then the following are equivalent

\(i\) \(A\) is a CSA over \(k\)

\(ii\) There are \(a, b \in k \setminus \{0\}\) and a \(k\)-basis \(\{1, i, j, k\}\) for \(A\) such that the multiplication on \(A\) is given by

\[
* i^2 = a \\
* j^2 = b \\
* ij = k = -ji
\]

**Proof.** We quickly sketch both directions

\(ii) \Rightarrow i)\) This is done by some explicit computations similar to the example of a matrix ring.

\(i) \Rightarrow ii)\) By the Wedderburn Theorem, \(A\) is either a matrix ring or a division algebra. In the first case we choose \(i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, a = -1, b = 1\)

for the second case we first note that for any \(x \in A\): \(1, x\) and \(x^2\) are necessarily linearly dependent. Next we construct a basis \(\{1, i', j', i'j'\}\) with \((i')^2 = a', (j')^2 = b'\) but where the last condition might fail. As a last step we can tweak this last basis in order for \(ij = -ji\) to hold.

**Remark.** It is known that in case \(k = \mathbb{R}\) every CSA of dimension 4 is isomorphic to either \(M_2(\mathbb{R})\) (with \(a = -1, b = 1\)) or \(\mathbb{H}\), the Hamilton quaternions (with \(a = b = -1\)). By Theorem 1.8, the latter must have a splitting field of dimension 2 over \(\mathbb{R}\) and indeed \(\mathbb{H} \otimes_\mathbb{R} \mathbb{C} \cong M_2(\mathbb{C})\).
2 Azumaya algebras over local rings

We now generalize CSAs over a field to Azumaya algebras over local rings. There are several equivalent ways to define Azumaya algebras. Following the book [Mil80] we start with the following rather technical definition:

**Definition 2.1.** Let \( R \) be a commutative local ring (this will be the case throughout this section) and let \( A \) be an associative \( R \)-algebra such that \( R \to A \) identifies \( R \) with a subring of \( Z(A) \) (i.e. the structure morphism is injective). Then \( A \) is called an Azumaya algebra over \( R \) is \( A \) is free of finite rank \( l \) as an \( R \)-module and if the following map is an isomorphism:

\[
\phi_A : A \otimes_R A^{\text{op}} \to \text{End}_R(A) : a \otimes a' \mapsto (x \mapsto axa')
\]

where \( A^{\text{op}} \) is the opposite algebra to \( A \) (i.e. the same additive structure and the multiplicative structure given by \( a \cdot b := b \cdot a \)).

**Remark.**

- As we require \( A \) to be free over \( R \), the inclusion \( R \subset Z(A) \) is automatic.
- \( \phi_A \) always is an \( R \)-algebra morphism, so only the bijectivity in the definition is a nontrivial condition.

In the case where \( R = k \) is a field we have the following:

**Proposition 2.2.** If \( A \) is a CSA over a field \( k \), then \( A \) is Azumaya over \( k \).

**Proof.** Let \( \dim_k(A) = l \) then \( \phi_A \) is a morphism between \( k \)-algebras which both have dimension \( l^2 \) over \( k \). Hence it suffices to check injectivity. Note that \( A \otimes_k A^{\text{op}} \) is a CSA over \( k \) hence \( \ker(\phi_A) = 0 \) or \( A \otimes_k A^{\text{op}} \). As the second option is obviously false we have proven injectivity of \( \phi_A \). \( \square \)

The other direction is also true and follows from the following proposition

**Proposition 2.3.** Let \( A \) be an Azumaya algebra over \( R \), then \( Z(A) = R \) and there is a bijection between the (two-sided) ideals of \( A \) and the ideals of \( R \):

\[
\begin{align*}
\{ \text{Ideals of } A \} & \overset{\cong}{\longrightarrow} \{ \text{Ideals of } R \} \\
\mathcal{I} & \mapsto \mathcal{I} \cap R \\
\mathcal{J}A & \leftrightarrow \mathcal{J}
\end{align*}
\]

**Proof.** Let \( \psi \in \text{End}_R(A) \) and \( c \in Z(A) \) then for all \( a \in A \) we have \( c\psi(a) = \psi(ca) = \psi(ac) = \psi(a)c \) because \( \psi \) is given by multiplication by elements in \( A \) as \( A \) is Azumaya. Similarly \( \psi(\mathcal{I}) \subset \mathcal{I} \) for each ideal \( \mathcal{I} \) of \( A \). Now let \( 1 = a_1, \ldots, a_l \) be a basis for \( A \) as an \( R \)-module and define \( \chi_i \in \text{End}_R(A) \) by \( \chi_i(a_j) = \delta_{ij} \). Write \( c = \sum_i r_i a_i \) with all \( r_i \in R \), then

\[
c = 1 \cdot c = \chi_1(a_1)c = \chi_1(a_1c) = \chi_1(1 \cdot c) = \chi_1 \left( \sum_{i=1}^l r_i a_i \right) = r_1 \in R
\]
Now we check the bijection between the sets of ideals. As the maps are well defined, it suffices to prove $I = (I \cap R)A$ and $J = JA \cup R$. Both equalities are trivial to check.

Corollary 2.4. An Azumaya algebra over a field is a CSA.

Proposition 2.5. Let $(R, m), (R', m')$ be commutative local rings and let $A$ be a free $R$-module of rank $l$. Assume there is a morphism $R \rightarrow R'$ then:

i) If $A$ is Azumaya over $R$ then $A \otimes_R R'$ is Azumaya over $R'$.

ii) If $A \otimes R/m$ is Azumaya (hence CSA) over $R/m$ then $A$ is Azumaya over $R$.

Proof. We have the following commutative diagram:

\[
\begin{array}{ccc}
\phi_A \otimes R' : (A \otimes_R A^{op}) \otimes_R R' & \xrightarrow{\cong} & \text{End}_R(A) \otimes_R R' \\
\phi_{A \otimes R'} : (A \otimes_R R') \otimes_{R'} (A \otimes_R R')^{op} & \xrightarrow{\cong} & \text{End}_{R'}(A \otimes_R R')
\end{array}
\]

The first statement follows immediately from this diagram. For the second statement note that surjectivity of $\phi_A \otimes_R R/m$ implies surjectivity of $\phi_A$ by Nakayama’s Lemma. For the injectivity we need a technical Lemma, e.g. [Mil80, Lemma IV.1.11]

Corollary 2.6. Let $A$ be a free module of rank $l$ over $(R, m)$ and let $k = R/m$, then the following are equivalent:

- $A$ is Azumaya over $R$
- $A \otimes k$ is a CSA over $k$
- $A \otimes \overline{k} \cong M_n(\overline{k})$

In particular $l = n^2$ for some $n \in \mathbb{N}$

Corollary 2.7. The tensor product of two Azumaya algebras is an Azumaya algebra.

- $M_n(R)$ is Azumaya over $R$

We now state the main result of this section

Proposition 2.8 (Skolem-Noether). Let $A$ be Azumaya over $R$, then every $\psi \in \text{Aut}_R(A)$ is inner. I.e. for any such $\psi$ there is a unit $u \in A^*$ such that $\psi(a) = uau^{-1}$. 

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Proof. Given $\psi \in Aut_R(A)$, there are two different ways to turn $A$ into an $A \otimes_R A^{op}$-module:

\[
\begin{align*}
(a_1 \otimes a_2)a &= a_1aa_2 \\
(a_1 \otimes a_2)a &= \psi(a_1)aa_2
\end{align*}
\]

Denote the resulting $A \otimes_R A^{op}$-modules by $A$, respectively $A'$.

Both $A'$ and $\overline{A}$ are simple $A \otimes_R A^{op}$-modules. This is based on the fact that $A \otimes_R A^{op}$-submodules of $\overline{A}$ or $\overline{A'}$ correspond to two-sided ideals of the central simple algebra $\overline{A}$ (the argument for $\overline{A'}$ uses the fact that $\psi$ is not just an endomorphism but an automorphism).

By Proposition 1.9, $\overline{A} \otimes_{\overline{R}} \overline{A}^{op}$ is a CSA over $\overline{R} = R/m$ and thus it is of the form $M_n(D)$ for some division algebra $D$ over $\overline{R}$. All simple modules over $M_n(D)$ are of the form $D^n$, so there must be an isomorphism of $A \otimes_R A^{op}$-modules:

\[
\chi : A \rightarrow \overline{A}
\]

We now claim that this lifts to a surjective $A \otimes_R A^{op}$-module morphism $\chi : A \rightarrow A'$.

First suppose the claim holds, then setting $u = \psi(1)$ gives:

\[
\psi(a)u = (a \otimes 1)u = \chi((a \otimes 1)1) = \chi((1 \otimes a)1) = (1 \otimes a)\chi(1) = ua
\]

Surjectivity of $\chi$ gives the existence of an $a_0 \in A$ such that $\chi(a_0) = 1$, hence

\[
1 = \chi(a_0) = \chi((1 \otimes a_0)1) = ua_0
\]

implying that $u$ is invertible in $A$.

Now we prove the claim: Note that we have the following diagram of $A \otimes_R A^{op}$-module morphisms:

\[
\begin{array}{ccc}
A & \rightarrow & End_R(A) \\
\downarrow & & \downarrow \\
\overline{A} & \rightarrow & \overline{A'}
\end{array}
\]

so the existence of $\chi$ follows if we can prove that $A$ is a projective $A \otimes_R A'$-module. As $A$ is free as an $R$-module there is an $R$-module morphism $g : A \rightarrow R$ such that $g(r) = r$. As $A$ is Azumaya we have $A \otimes_R A^{op} \cong End_R(A)$ and $A$ is a direct summand of $End_R(A)$ via

\[
A \xrightarrow{a \mapsto (a' \mapsto g(a')a)} End_R(A) \xrightarrow{f \mapsto (f(1))} A
\]

Finally surjectivity of $\chi$ follows from Nakayama’s Lemma. □
3 Azumaya algebras over schemes

Throughout this section, let $X$ be a locally Noetherian scheme.

**Proposition 3.1.** Let $A$ be an $O_X$-algebra of finite type as an $O_X$-module. Then the following are equivalent:

i) $A$ is a locally free $O_X$ module and $\phi_A : A \otimes A^{op} \to \text{End}_{O_X}(A)$ is an isomorphism

ii) $A_x$ is Azumaya over $O_{X,x}$ for each point $x$ in $X$

iii) $A_x$ is Azumaya over $O_{X,x}$ for each closed point $x$ in $X$

iv) $A$ is a locally free $O_X$ module and $A_x \otimes O_{X,x}/m_x$ is a CSA over $k(x) = O_{X,x}/m_x$

v) There is a covering $(U_i \to X)$ for the étale topology such that for each $i : A \otimes_{O_X} O_{U_i} \cong M_{r_i}(O_{U_i})$ for some $r_i$

vi) There is a covering $(U_i \to X)$ for the fppf (flat and of finite presentation) topology such that for each $i : A \otimes_{O_X} O_{U_i} \cong M_{r_i}(O_{U_i})$ for some $r_i$

**Remark.** Note that in the above proposition we made some abuse of notation: if $(U_i \to X)$ is a covering in the étale or flat topology, then $A \otimes_{O_X} O_{U_i}$ is short hand notation for $f_i^* A$.

**Proof.** We prove $i) \iff ii) \iff iii)$ and $ii) \Rightarrow v) \Rightarrow vi) \Rightarrow iv) \Rightarrow ii)$.

i) $\iff ii)$ As $A$ is locally free we have $(A \otimes_{O_X} A^{op})_x = A_x \otimes_{O_{X,x}} (A_x)^{op}$ and $\text{End}_{O_X}(A)_x = \text{End}_{O_{X,x}}(A_x)$. And $\phi_A$ is an isomorphism if and only if all $(\phi_A)_x = \phi_{A_x}$ are isomorphisms.

ii) $\Rightarrow iii)$ obvious

iii) $\Rightarrow ii)$ If $\eta$ is some non-closed point, then there is a closed point $x \in \eta$ (on a locally Noetherian scheme we have existence of closed points: [Sta15, Tag 01OU, Lemma 27.5.9]). Then there is a (non-local) morphism of local rings $O_{X,x} \to \tilde{k}(x)$ with separable closure $\tilde{k}(x)$ and $k(x) = O_{X,x}/m_x$.

Let $x$ be a point in $X$, let $k(x) = O_{X,x}/m_x$ with separable closure $\tilde{k}(x)$ and let $\tau : \text{Spec}(\tilde{k}(x)) \to X$ be the associated geometric point. Recall from the previous lecture that the étale stalk $O_{X,\tau}$ is a module over the Zariski stalk $O_{X,x}$, because it is its strict Henselization. In particular we can apply Proposition 2.5 to obtain that $A_x \otimes O_{X,\tau}$ is Azumaya over the strict Henselian (local) ring $O_{X,\tau}$. The residue field of $O_{X,\tau}$ is given by $\tilde{k}(x)$ and hence is separably closed. By Theorem 1.8, this implies that $A_x \otimes \tilde{k}(x)$ is split. Then $A_x \otimes O_{X,\tau} \cong M_{r_i}(O_{X,\tau})$. From this it follows that there is an étale morphism $U \to X$ whose image contains $x$ such that $f^* A \cong M_{r_i}(U)$. 

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(Details: as \( A_x \otimes O_{X,x} \cong M_r(O_X) \) there is some basis \((e_{ij})_{i,j=1,...,r}\) (suggestive notation!) for \( A_x \otimes O_{X,x} \) as an \( O_{X,x} \) module such that the multiplicative relations are given by \( e_{ij}e_{jl} = e_{il} \). As \( A \) is locally free (in the Zariski and hence also étale topology) we have \( A_x \otimes O_{X,x} \cong A_x \) and there must exist some étale environment \( V \) of \( x \) such that there is a basis \((c_{ij})_{i,j=1,...,r}\) for \( A(V) \) which \( (c_{ij})_x = e_{ij} \). (The fact that the \( c_{ij} \) give a basis for \( A(V) \) is heavily based on the locally freeness of \( A \).)

For each \( i,j,l \) the relation \( (c_{ij})_x \cdot (c_{jl})_x = (c_{il})_x \) implies that there is some étale environment \( V_{ijl} \to V \to X \) for which \( c_{ij}|_{V_{ijl}} \cdot c_{jl}|_{V_{ijl}} = c_{il}|_{V_{ijl}} \).

\[
\implies v)
\]

Trivial as each étale morphism is flat and of finite presentation.

\[
v) \Rightarrow iv)
\]

Let \( (U_i \xrightarrow{f_i} X) \) be a covering for the flat topology such that for each \( i : A \otimes O_{U_i} \cong M_r(O_{U_i}) \). We first check that \( A \) is locally free. Let \( U = \coprod U_i \), then \( \coprod f_i : U \to X \) is surjective and flat (hence by definition faithfully flat). In particular the fact that \( f^*A \) is a flat \( O_U \)-module (this is true because it is finitely generated and locally free) implies that \( A \) is a flat and hence locally free \( O_X \)-module.

(Details: suppose we are given a short exact sequence of \( O_X \)-modules:

\[
0 \to F \to G \to H \to 0
\]

As \( f \) is flat the following is exact as well:

\[
0 \to f^*F \to f^*G \to f^*H \to 0
\]

Now \(- \otimes_{O_U} f^*A\) is an exact functor pullback commutes with tensor product, so we get the exact sequence

\[
0 \to f^*((F \otimes_{O_X} A) \to f^*(G \otimes_{O_X} A) \to f^*(H \otimes_{O_X} A) \to 0
\]

and as \( f \) is faithfully flat this implies that

\[
0 \to F \otimes_{O_X} A \to G \otimes_{O_X} A \to H \otimes_{O_X} A \to 0
\]

is itself exact and hence \( A \) is a flat \( O_X \)-module.)

Next let \( x \in X \) be chosen at random. We check that \( A_x \otimes_{O_{X,x}} k(x) \) is a CSA over \( k(x) \). Take \( i \) such that \( x \) is in the image of \( f_i \), say \( f_i(y) = x \).
for some \( y \in U_i \). Then \((f^*_i \mathcal{A})_y = \mathcal{A}_x \otimes_{\mathcal{O}_{U_i,y}} \mathcal{O}_{U_i,y}\) and thus
\[
\begin{align*}
(\mathcal{A}_x \otimes_{\mathcal{O}_{U_i,x}} k(x)) \otimes_{k(x)} k(y) &\cong \mathcal{A}_x \otimes_{\mathcal{O}_{U_i,x}} \left( k(x) \otimes_{k(x)} k(y) \right) \\
&\cong \mathcal{A}_x \otimes_{\mathcal{O}_{U_i,x}} (\mathcal{O}_{U_i,y} \otimes \mathcal{O}_{U_i,y}) k(y) \\
&\cong (f^*_i \mathcal{A})_y \otimes \mathcal{O}_{U_i,y} k(y) \\
&\cong (f^*_i \mathcal{A})_y \otimes \mathcal{O}_{U_i,y} k(y) \\
&\cong (f^*_i \mathcal{A})_y \otimes \mathcal{O}_{U_i,y} k(y)
\end{align*}
\]

This implies that some base change of \( \mathcal{A}_x \otimes k(x) \) is a CSA and hence \( \mathcal{A}_x \otimes k(x) \) is itself a CSA.

\( iv \Rightarrow ii \) This was already proved in Proposition 2.5.

We end this seminar by the following version of the Skolem-Noether theorem:

**Proposition 3.2.** Let \( \mathcal{A} \) be Azumaya on \( X \), then every \( \psi \in \text{Aut}(\mathcal{A}) \) is locally, for the Zariski topology on \( X \) an inner automorphism. I.e. there is a Zariski-open covering \( (U_i \to X) \) such that \( \psi|_{U_i} \) is of the form \( a \mapsto uau^{-1} \) for some \( u \in \Gamma(U_i, \mathcal{A})^* \).

**Proof.** Let \( x \in X \), then by the Skolem-Noether Theorem for local rings (Proposition 2.8) there is a \( u_x \in \mathcal{A}^*_x \) such that \( \psi_x(a_x) = u_x^{-1} a_x u_x \) for all \( a_x \in \mathcal{A}_x \). As \( u_x \) is invertible in \( \mathcal{A}_x \), there is some open environment \( U \) of \( x \) such that \( u_x \) is given by the fibre of some invertible \( u \in \Gamma(U, \mathcal{A})^* \). (by definition \( u_x \) being invertible asks for \( u_x v_x = v_x u_x = 1_x \). Again by definition this implies that there are section \( u, v \) on some environment \( U \) of \( x \) such that \( uv = vu = 1_U \) and \( (u)_x = u_x, (v)_x = v_x \).) As \( \mathcal{A} \) is locally free, we may assume \( \mathcal{A} \) is free on \( U \) (shrinking \( U \) if necessary). I.e. there exist \( a_1, \ldots, a_{n^2} \in \Gamma(U, \mathcal{A}) \) which give a basis for \( \mathcal{A}|_U \) as a free \( \mathcal{O}_U \)-module. We now have two morphisms of \( \mathcal{O}_U \)-modules
\[
\psi|_U : \mathcal{A}_U \to \mathcal{A}_U \\
\varphi : \mathcal{A}_U \to \mathcal{A}_U : a \mapsto u^{-1} au
\]
whose stalks agree at \( x \). I.e. \( (\psi|_U(a_i))_x = (\varphi(a_i))_x \) for all \( i = 1, \ldots, n^2 \). By again shrinking \( U \) if necessary we can assume \( \psi|_U(a_i) = \varphi(a_i) \) for all \( i \) and hence \( \psi|_U \) is inner. \( \Box \)

**References**

