Stacks

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1 Introduction

We'll introduce *stacks*, which are a generalization of the notion of sheaves. They are useful, informally speaking, whenever objects are considered *upto isomorphism*. More precisely, stacks keep track of how objects *and morphisms between them* glue, while sheaves only care about the objects.

A sheaf $\mathcal{F}$ associates to each U in the site a set $\mathcal{F}(U)$, maybe with abelian group structure or whatever. So naively we could just take $\mathcal{F}(U)$ to be a category, and $\mathcal{F}(U) \to \mathcal{F}(V)$ a restriction *functor* for every map $V \to U$ of the site. But how to translate for example the notion of isomorphism of sheaves? We have the problem that isomorphism of categories is here a *useless concept*.

Example 1. Let A and A' be two isomorphic rings (e.g. $\mathbb{C}$ and $\mathbb{R}[x]/x^2+1$). Then there is no obvious isomorphism from the category of A-modules to the category of A'-modules. If we consider for example the functors

$$\begin{tikzcd} \mathsf{Mod}(A) \arrow[r, bend left, "A' \otimes_A -"] & \mathsf{Mod}(A') \arrow[l, bend left, "A \otimes_{A'} -"] \end{tikzcd}$$

then these are not mutually inverse, because for instance $A' \otimes_A A$ is only *isomorphic* to A' and not really *equal*. In fact, it would be difficult to define what we mean by isomorphism, because the objects of a category almost never form a set.

The alternative is to look at *equivalence* of categories. Recall that two categories $\mathcal{C}$ and $\mathcal{D}$ are called equivalent if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with $FG \simeq 1_D$ and $GF \simeq 1_C$. This automatically brings us into the realm of 2-categories. So we will start our exposition on stacks by recalling some technicalities from category theory.

2 Some category theory

Definition 2. A *monoidal category* is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object I in $\mathcal{C}$, equipped with natural isomorphisms

$$\begin{aligned} (A \otimes B) \otimes C \simeq A \otimes (B \otimes C) \qquad & \text{for all } A, B, C \text{ in } \mathcal{C} \\ A \otimes I \simeq A \simeq I \otimes A \qquad & \text{for all } A \text{ in } \mathcal{C} \end{aligned}$$

satisfying some coherency conditions. The functor $\otimes$ is called the *tensor product* and I is called the *unit*.

Example 3.

- The category of bimodules over a ring A, with tensor product $\otimes_A$ and unit A.
- The category of left modules (or right modules, or symmetric bimodules) over a commutative ring A, with tensor product $\otimes_A$ and unit A.
- The category of sets with tensor product $\times$ and unit some set with one element.
- The category of categories with tensor product $\times$ and unit some category with one object and only the identity morphism.

Definition 4. Let $(\mathcal{A}, \otimes, I)$ be a monoidal category. A category *enriched in* $\mathcal{A}$ is a collection $\mathcal{C}$ equipped with a so-called *Hom object*

$$\mathrm{Hom}(M, N) \text{ in } \mathcal{A}$$

for all M, N in $\mathcal{C}$. These Hom objects have to be accompanied by a composition rule

$$\mathrm{Hom}(M, N) \otimes \mathrm{Hom}(N, P) \to \mathrm{Hom}(M, P)$$

for all M, N and P in $\mathcal{C}$, and an identity morphism

$$I \to \mathrm{Hom}(M, M)$$

for every M in $\mathcal{C}$. The composition is required to be associative and composition with the identity morphism has to act trivially on the Hom objects (on both sides). Formally, the latter conditions give diagrams involving the associativity and unit isomorphisms of $\mathcal{A}$ (however in practice these are often considered only implicitly).

Example 5.

- Like any abelian category, the category of modules over a ring A is enriched over the category of abelian groups.
- Locally small categories are just categories enriched in sets.
- The category of (small) categories is enriched in categories.

More generally, we have the following definition.

Definition 6. A *2-category* is a category enriched in categories.

So instead of a *set* of morphisms, we have a *category* of morphisms, in particular we have "morphisms between morphisms", which we will call 2-*morphisms*. Unraveling above definition, we see that there are two kinds of compositions of 2-morphisms: a *vertical composition* of

$$\begin{tikzcd} M \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=C}] \arrow[r, bend right=60, ""{name=D, above}] & N \arrow[Rightarrow, from=U, to=C, "\alpha"] \arrow[Rightarrow, from=C, to=D, "\beta"] \end{tikzcd}$$

denoted by $\beta \circ \alpha$ and a *horizontal composition* of

$$\begin{tikzcd} M \arrow[r, bend left, ""{name=A, below}] \arrow[r, bend right, ""{name=B, above}] & N \arrow[r, bend left, ""{name=C, below}] \arrow[r, bend right, ""{name=D, above}] & P \arrow[Rightarrow, from=A, to=B, "\alpha"] \arrow[Rightarrow, from=C, to=D, "\gamma"] \end{tikzcd}$$

denoted by $\gamma\alpha$. From the functoriality of the composition, we get for 2-morphisms

$$\begin{tikzcd} M \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & N \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=3ex, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3ex, "\Downarrow\delta"] & P \end{tikzcd}$$

that $(\delta \circ \gamma)(\beta \circ \alpha) = \delta\beta \circ \gamma\alpha$.

Example 7.

- The category of (small) categories is a 2-category, with natural transformations as 2-morphisms.
- A *groupoid* is a small category in which every morphism is an isomorphism. So the category of groupoids inherits a 2-category structure from the category of small categories.

3 Categories fibered in groupoids

This part is strongly based on the sections [Stacks, Tag 02XJ,Tag 003S,Tag 02XU].

Consider categories $\mathcal{X}$ and $\mathcal{C}$ together with a functor $p : \mathcal{X} \to \mathcal{C}$. You should think of $\mathcal{X}$ as a category *lying over* $\mathcal{C}$; p is then the functor that associates to each object of $\mathcal{X}$ its "base" in $\mathcal{C}$.

Definition 8. A morphism $\phi : y \to x$ in $\mathcal{X}$ is said to be *strongly cartesian* if it satisfies the following universal property. For every $f : z \to x$ and $g : p(z) \to p(y)$ such that $p(\phi)g = p(f)$ there is a unique $h : z \to y$ with $\phi h = f$ and $p(h) = g$.

Lemma 9.

- *The composition of strongly cartesian morphisms is again strongly cartesian.*
- *Isomorphisms are strongly cartesian.*
- *If $f : y \to x$ is strongly cartesian, then it is an isomorphism if and only if $p(f)$ is an isomorphism.*

Notice that the definition is very similar to the definition of a pullback. This "pulling back" of elements in $\mathcal{X}$ is going to replace the restriction morphism that we had for sheaves. So if we want to imitate presheaves, we should ask that every element can be pulled back.

Definition 10. Let $p : \mathcal{X} \to \mathcal{C}$ be a functor. We say that $\mathcal{X}$ is *fibered* over $\mathcal{C}$ if for every x in $\mathcal{X}$ and $g : s \to p(x)$ in $\mathcal{C}$, there is a strongly cartesian morphism $f : y \to x$ with $p(y) = s$ and $p(f) = g$. In this case, the *fiber* over an object s of $\mathcal{C}$ is the subcategory consisting of the objects x of $\mathcal{X}$ with $p(x) = s$ and the morphisms lifting id_s. A fibered category $p : \mathcal{X} \to \mathcal{C}$ is called *fibered in groupoids* if all fibers are groupoids (i.e. all morphisms in the fiber are isomorphisms).

If we look specifically at categories fibered in *groupoids*, then the situation simplifies as follows.

Proposition 11. *Let $\mathcal{X}$ be a category fibered in groupoids over $\mathcal{C}$. Then every morphism in $\mathcal{X}$ is strongly cartesian.*

Proof. Take a morphism $z \to x$ in $\mathcal{X}$. There is a strongly cartesian morphism $y \to x$ with $p(y) = p(z)$. By definition of strongly cartesian, the identity map $p(z) \to p(y)$ and the morphism $z \to x$ together give a morphism $z \to y$, which is an isomorphism because z and y live in the same fiber. □

Definition 12. Let $\mathcal{X}$ and $\mathcal{X}'$ be categories fibered in groupoids over $\mathcal{C}$, with structure morphisms p resp. p'. A *morphism* $\mathcal{X} \to \mathcal{X}'$ is a functor $F : \mathcal{X} \to \mathcal{X}'$ satisfying $p'F = p$. Let F and G be two morphisms from $\mathcal{X}$ to $\mathcal{X}'$. Then a *2-morphism* $\alpha : F \Rightarrow G$ is a natural transformation such that the induced natural transformation from $p'F = p$ to $p'G = p$ is the identity.

After writing down the 2-morphisms explicitly, it becomes obvious that they are all invertible. It is also easy to prove that the above definition really gives a 2-category, because it is closely related to the 2-category of categories.

Let $F : \mathcal{X} \to \mathcal{X}'$ be a morphism of categories fibered in groupoids over $\mathcal{C}$. Then F is an equivalence if there is a morphism $G : \mathcal{X}' \to \mathcal{X}$ such that FG and GF are both the identity upto 2-isomorphism. Note that this definition works for any 2-category. Moreover, this notion is a priori different from saying that F is just an equivalence of categories from $\mathcal{X}$ to $\mathcal{X}'$. We can however prove that these notions agree in our case.

There is a functor

$$\mathsf{PSh}(\mathcal{C}) \longrightarrow \mathsf{CFG}(\mathcal{C})$$

from presheaves over $\mathcal{C}$ to categories fibered in groupoids over $\mathcal{C}$, associating to each presheaf $\mathcal{F}$ the following category:

- The objects are couples (U, x) with U in $\mathcal{C}$ and $x \in \mathcal{F}(U)$.
- The morphisms $(V, y) \to (U, x)$ are maps $f : V \to U$ such that x restricts to y along f.

It is easy to see that this is indeed a category fibered in groupoids, and (as advertised) it is also functorial: a morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ induces a 1-morphism between the corresponding categories by sending (U, x) to $(U, \phi(x))$ and $f : V \to U$ to itself. The latter yields a map $(V, \phi(y)) \to (U, \phi(x))$ because $\phi(x)$ restricts again to $\phi(y)$.

Note that the above functor is fully faithful, so from now on we will mean with "presheaf" a category fibered in groupoids that is in the image of this functor. Take further $\mathcal{X}$ any category fibered in groupoids over $\mathcal{C}$, such that objects in the fibers of $\mathcal{X}$ only have trivial automorphisms (on the Stacks project, such $\mathcal{X}$ is said to be fibered in *setoids*). Then it can be proved that $\mathcal{X}$ is equivalent to a unique presheaf. Moreover, if $\mathcal{X}$ and $\mathcal{Y}$ are both equivalent to presheaves, then maps between them are (upto 2-iso) just given by maps between the corresponding presheaves.

Definition 13. A category fibered in groupoids is called *representable* if it is equivalent to a representable presheaf.

For an object U of $\mathcal{C}$, we will denote the corresponding representable presheaf by h_U.

Theorem 14 (2-Yoneda Lemma)**.** *Let $\mathcal{X}$ be a category fibered in groupoids over $\mathcal{C}$, and U and object of $\mathcal{C}$. There is an equivalence of categories*

$$\mathrm{Hom}(h_U, \mathcal{X}) \to \mathcal{X}_U$$

where $\mathcal{X}_U$ denotes the fiber of $\mathcal{X}$ over U.

Proof. See [Stacks, Tag 004B]. □

Let $\mathcal{X}$ be a category fibered in groupoids over $\mathcal{C}$. Using the axiom of choice, we can choose for every $V \to U$ in $\mathcal{C}$ and x in $\mathcal{X}$ some preferred strongly cartesian morphism $f^*x \to x$. This f^* automatically becomes a functor.

Definition 15. A category fibered in groupoids, is called *split* if we can choose preferred strongly cartesian morphisms such that

$$g^* f^* = (fg)^*$$

for all morphisms f and g in $\mathcal{C}$.

Proposition 16. *Every category fibered in groupoids is equivalent to a split one.*

Proof. See [Stacks, Tag 004A]. □

4 Stacks

Presheaves are called sheaves whenever local sections that agree on intersections, can be glued to a unique global section. Stacks are a generalization of sheaves, for which the "sections" do not need to be equal on intersections, but only isomorphic. However, these isomorphisms should be compatible, a criterion that is expressed by cocycles.

This part is based on the sections [Stacks, Tag 0268, Tag 02ZH]. We fix a category fibered in groupoids $\mathcal{X}$ over a *site* $\mathcal{C}$. We can and will assume that $\mathcal{X}$ is split.

For a covering $W \to U$ we look at the maps

$$W \times_U W \times_U W \;\begin{matrix} \xrightarrow{\mathrm{pr}_{12}} \\ \xrightarrow{\mathrm{pr}_{02}} \\ \xrightarrow{\mathrm{pr}_{01}} \end{matrix}\; W \times_U W \;\begin{matrix} \xrightarrow{\mathrm{pr}_1} \\ \xrightarrow{\mathrm{pr}_0} \end{matrix}\; W \xrightarrow{\mathrm{pr}} U,$$

and we use the (nonstandard) notation

$$\begin{aligned} \mathrm{prr}_0 &:= \mathrm{pr}_0 \mathrm{pr}_{02} = \mathrm{pr}_0 \mathrm{pr}_{01} \\ \mathrm{prr}_1 &:= \mathrm{pr}_0 \mathrm{pr}_{12} = \mathrm{pr}_1 \mathrm{pr}_{01} \\ \mathrm{prr}_2 &:= \mathrm{pr}_1 \mathrm{pr}_{12} = \mathrm{pr}_1 \mathrm{pr}_{02} \,. \end{aligned}$$

Let U be an object of the site, and consider the trivial covering $U \to U$. Let x be an element in $\mathcal{X}_U$. Then we get a canonical isomorphism

$$\phi : \mathrm{pr}_1^* x \to \mathrm{pr}_0^* x$$

inducing a commutative diagram

$$\begin{tikzcd} \mathrm{prr}_1^* x \arrow[dd, bend right, "\mathrm{pr}_{01}^* \phi"'] & \\ & \mathrm{prr}_2^* x \arrow[ul, bend right, "\mathrm{pr}_{12}^* \phi"'] \arrow[dl, bend left, "\mathrm{pr}_{02}^* \phi"] \\ \mathrm{prr}_0^* x & \end{tikzcd} \quad . \tag{1}$$

Now consider another covering $f : V \to U$ (to avoid confusion arising from index sets, we assume that we can replace any covering by a singleton covering, take for example U quasi-compact). We can pull back ϕ along f to get an isomorphism

$$f^*\phi : \mathrm{pr}_1^* x \to \mathrm{pr}_0^* x$$

for the covering f, for which we also get a commutative diagram as in (1). This isomorphism will be called the *canonical descent datum* for the covering. More generally, we will call a morphism

$$\mathrm{pr}_1^* x \to \mathrm{pr}_0^* x$$

a *descent datum* if the induced diagram as in (1) is commutative. A descent datum is called *effective* if it is the pullback of a canonical descent datum along an isomorphism.

Definition 17. The category fibered in groupoids $\mathcal{X}$ is called a *stack* over the site $\mathcal{C}$ if

- for each U in $\mathcal{C}$ and x, y in $\mathcal{X}_U$, $\mathrm{Hom}(x, y)$ is a sheaf on U, and
- all descent data (for all coverings) are effective.

Note that a presheaf is a stack if and only if it is a sheaf. In particular, representable presheaves can be seen as stacks, when working in a subcanonical topology.

5 The inertia sheaf on a stack

Using the 2-Yoneda Lemma, we can interpret each stack $\mathcal{X}$ as a comma category $\mathcal{C}/\mathcal{X}$. On this comma category we have a Grothendieck topology, inherited from the topology on $\mathcal{C}$. If $\mathcal{C}$ is the big site of a scheme for a certain topology, then $\mathcal{C}/\mathcal{X}$ becomes in this way the corresponding big site for $\mathcal{X}$. Now sheaves on $\mathcal{X}$ are just sheaves on this site. For further remarks, see [Stacks, Tag 06TN].

Definition 18. Let $\mathcal{X}$ be a fibered category. The *inertia category* $\mathcal{I}_{\mathcal{X}}$ is the category with

- as objects the pairs (x, α) with x in $\mathcal{X}$ and $\alpha : x \to x$ an automorphism lifting the identity of $p(x)$, and
- as morphisms $(x, \alpha) \to (y, \beta)$ the commutative diagrams

$$\begin{tikzcd} x \arrow[r, "\phi"] \arrow[d, "\alpha"'] & y \arrow[d, "\beta"] \\ x \arrow[r, "\phi"'] & y \end{tikzcd} .$$

Note that we have a morphism $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$, by sending (x, α) to x and ϕ to itself (using the above notations). Using the definition of strongly cartesian morphisms, one can calculate that this turns $\mathcal{I}_{\mathcal{X}}$ into a category fibered in groupoids over $\mathcal{X}$.

It will even arise from a presheaf on $\mathcal{X}$. Indeed, if we look at some x in $\mathcal{X}$ and the corresponding fiber in $\mathcal{I}_{\mathcal{X}}$, then the objects in this fiber are the automorphisms of x, and the morphisms between automorphisms α and β are given by commutative diagrams

$$\begin{tikzcd} x \arrow[r, "\phi"] \arrow[d, "\alpha"'] & x \arrow[d, "\beta"] \\ x \arrow[r, "\phi"'] & x \end{tikzcd},$$

with the extra condition that ϕ should go to the identity morphism in $\mathcal{X}$. But this means that ϕ is itself an identity morphism. This implies that $\mathcal{I}_{\mathcal{X}}$ actually arises from a presheaf on $\mathcal{X}$.

Now if $\mathcal{C}$ is equipped with a Grothendieck topology and $\mathcal{X}$ is a stack, then we want to show that $\mathcal{I}_{\mathcal{X}}$ is actually a *sheaf* over $\mathcal{X}$. If you write this down explicitly, then this follows by the definition of a stack (the condition that the hom-sets should give a sheaf).

6 Gerbes

Definition 19. Let $\mathcal{X}$ be a stack on a site $\mathcal{C}$. Then $\mathcal{X}$ is called a *gerbe* if

- for every U in $\mathcal{C}$ there is a covering $U' \to U$ for which $\mathcal{X}_{U'}$ is non-empty, and
- for every U and every x, y in $\mathcal{X}_U$ there is a covering $U' \to U$ such that x and y become isomorphic over U'.

Further, let A be a sheaf of abelian groups on $\mathcal{C}$. Then an A-*gerbe* is a gerbe equipped with an isomorphism of sheaves $A_{\mathcal{X}} \to \mathcal{I}_{\mathcal{X}}$.

The above definition is the one from [Lie04], and is only valid for sheaves of abelian groups.

Note that for every sheaf F on $\mathcal{X}$ we have an action

$$F \times \mathcal{I}_{\mathcal{X}} \to F \tag{2}$$

given by pulling back sections of F along the elements of $\mathcal{I}_{\mathcal{X}}$. If $\mathcal{X}$ is an A-gerbe, then this gives an action

$$F \times A_{\mathcal{X}} \to F. \tag{3}$$

References

[Lie04] M. Lieblich. “Moduli of twisted sheaves”. In: *ArXiv Mathematics e-prints* (2004). eprint: `math/0411337`.

[Stacks] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`. 2015.