Cell decompositions and flag manifolds

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The main goal of these notes is to give a description of the cohomology of a Grassmannian (as an abelian group), and generalise this to complete flag varieties. We mainly follow [Kre07], other useful references are given at the end.

1 Cell decompositions

Definition 1.1. A *stratification* of a scheme X is a decomposition $X = \bigsqcup_{j=1}^{n} C_j$, where each C_j is locally closed and $\overline{C_j} \setminus C_j$ is a disjoint union of some other C_i's. These C_j are called *strata* and their closures $\overline{C_j}$ are called the *closed strata*.

Example 1.2. Let G be a connected algebraic group over $\mathbb{C}$, acting on a variety V over $\mathbb{C}$. Then the orbits form a stratification of V.

Definition 1.3. A stratification is called *affine* if each stratum is isomorphic to some $\mathbb{A}^k$. Affine stratifications are also called *cell decompositions*.

Example 1.4.

- $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$ gives a cell decomposition of $\mathbb{P}^n$
- More generally, Grassmannians and flag varieties have cell decompositions (see later)

Theorem 1.5 (Example 19.1.11 in [Ful84])**.** *Let X be a scheme that admits a cell decomposition. Then $\gamma_X : \mathrm{A}_*(X) \to \mathrm{H}_*^{\mathrm{bm}}(X)$ is an isomorphism (where $\mathrm{H}_*^{\mathrm{bm}}$ denotes Borel–Moore homology).*

Remark 1.6. If X is a compact variety over $\mathbb{C}$, then $\mathrm{H}_*^{\mathrm{bm}}(X) \cong \mathrm{H}_{2*}(X)$, so in this case $\mathrm{A}_k(X) \cong \mathrm{H}_{2k}(X)$ is the free abelian group on the k-dimensional cells (i.e. real dimension $2k$). This is because usual homology can be computed using the cellullar complex, and in this case the differentials all vanish. It turns out that the ring structures of A_* and H_{2*} agree too.

Sketch of proof of theorem. For Borel-Moore homology we have a long exact sequence $\cdots \to \mathrm{H}_k^{\mathrm{bm}}(Y) \to \mathrm{H}_k^{\mathrm{bm}}(X) \to \mathrm{H}_k^{\mathrm{bm}}(U) \to \ldots$, where $Y \subset X$ is a closed subscheme and $U = X \setminus Y$. We also saw in the second week of the seminar that we have an exact sequence $\mathrm{A}_k(Y) \to \mathrm{A}_k(X) \to \mathrm{A}_k(U) \to 0$. By naturality of γ and diagram chasing, we can conclude that if γ_Y and γ_U are isomorphisms, then γ_X is too. By using induction, we can then show that γ_X is an isomorphism if X admits a cell decomposition (verify that $\gamma_{\mathbb{A}^k}$ is an isomorphism for each k). $\square$

2 Schubert cells

2.1 Definitions

We view the Grassmannian $\mathrm{G}(k,n)$ as the quotient $\mathrm{Mat}_{k\times n}^{\mathrm{rank}=k} / \mathrm{GL}(k)$ with action multiplication on the left (= row operations).

Definition 2.1. We define a *Schubert symbol* to be a sequence $1 \leq j_1 < j_2 < \cdots < j_k \leq n$.

To each Schubert symbol J corresponds a square matrix consisting of the rows given by J, and its determinant is the Plücker coordinate corresponding to J. Recall that the Plücker coordinates together constitute a map

$$\mathrm{G}(k,n) \to \mathbb{P}^{\binom{n}{k}-1},$$

called the Plücker embedding. This map is well-defined. Indeed, if $g \in \mathrm{GL}(k)$ and $M \in \mathrm{Mat}_{k\times n}^{\mathrm{rank=k}}$, then the Plücker coordinates of gM are $p_i' = \det(g)p_i$, where the p_i are the Plücker coordinates of M. This shows that gM and M are send to the same point in $\mathbb{P}^{\binom{n}{k}-1}$.

Lemma 2.2 (Gauss elimination)**.** *Each $M \in \mathrm{G}(k,n)$ has a unique representative matrix in row echelon form, i.e. we have a Shubert symbol J such that the corresponding square matrix is*

$$\begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

and such that to the right of the 1*'s coming from this submatrix are only zeroes.*

Example 2.3. The matrices

$$\begin{pmatrix} * & 0 & * & 0 & 1 & 0 \\ * & 0 & * & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where each $*$ is an arbitrary complex number, are all in row echelon form corresponding to the same Schubert symbol $(2,4,5)$.

Definition 2.4. The *Schubert cell* corresponding to J is defined as the variety $X_J^0 = \{M \in \mathrm{G}(k,n) |$ row echelon form of M has Schubert symbol $J\}$.

It is clear that the X_J^0 are locally closed subvarieties isomorphic to $\mathbb{C}^d$ with

$$d = i_1 + \cdots + i_k - \frac{k(k+1)}{2},$$

and that $\mathrm{G}(k,n) = \bigsqcup_J X_J^0$.

Definition 2.5. The *Schubert variety* corresponding to J is $X_J = \overline{X_J^0}$.

The Schubert variety X_J is always defined by the vanishing of determinants of some minors (this follows from the Gauss elimination algorithm). This corresponds to vanishing of some Plücker coordinates, namely the ones corresponding to Schubert cells I with $I \not\leq J$, where we use the following ordering on the Schubert symbols:

Definition 2.6. For two Schubert symbols $I : i_1 < \cdots < i_k$ and $J : j_1 < \cdots < j_k$ we define a partial ordering with $I \leq J$ if and only if $i_t \leq j_t$ for all t.

Proposition 2.7.

$$X_J^0 = \{\Sigma \subset \mathbb{C}^n \text{ } k\text{-dimensional} \mid \dim(\Sigma \cap \mathbb{C}^i) = \#\{1, \ldots, i\} \cap J\}$$
$$X_J = \{\Sigma \subset \mathbb{C}^n \text{ } k\text{-dimensional} \mid \dim(\Sigma \cap \mathbb{C}^i) \geq \#\{1, \ldots, i\} \cap J\}$$

Proof. For the first statement, look at the row echelon form of Σ. Then $\Sigma \in X_J^0 \Leftrightarrow \dim(\Sigma \cap \mathbb{C}^i) = \#1, \ldots, i \cap J$. For the second one, use the description by Plücker coordinates (or geometric intuition). $\square$

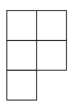
Corollary 2.8. $X_J = \bigsqcup_{I \leq J} X_I^0$*, in particular the Schubert cells form an affine stratification.*

Example 2.9. Consider the Schubert varieties in $\mathrm{G}(2, 4)$:

$X_\emptyset = \begin{pmatrix} * & * & 0 & 1 \\ * & * & 1 & 0 \end{pmatrix}$	all $\Sigma \subset \mathbb{C}^4$	all lines in $\mathbb{P}^3$
$X_1 = \begin{pmatrix} * & 0 & * & 1 \\ * & 1 & 0 & 0 \end{pmatrix}$	$\Sigma \cap \mathbb{C}^2$ is a line	lines incident to given line
$X_{11} = \begin{pmatrix} * & 0 & 1 & 0 \\ * & 1 & 0 & 0 \end{pmatrix}$	$\Sigma \subset \mathbb{C}^3$	lines in a plane
$X_2 = \begin{pmatrix} 0 & * & * & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{C}^1 \subset \Sigma$	lines through a point
$X_{21} = \begin{pmatrix} 0 & * & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{C}^1 \subset \Sigma \subset \mathbb{C}^3$	lines in a plane through a point
$X_{22} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\Sigma = \mathbb{C}^2$	line coinciding with given line

Note that the dimension of the Schubert variety/cell is the number of $*$'s. Also, there's an abuse of notation in the first column: the Schubert variety is the closure of the mentioned set of matrices, not equal. But this shouldn't be too ambiguous because there's a 1:1-correspondence.

Here we use the notation σ_λ where λ ranges over the Young diagrams fitting into a $k \times (n-k)$ rectangle. For example the Young diagram corresponding to $\lambda = (2, 2, 1)$ is



and the smallest rectangle it could fit in is a 3×2 one.

For a given Young diagram $\lambda = (\lambda_1, \ldots, \lambda_n)$, the corresponding Schubert symbol is given by $j_i = n - k + 1 - \lambda_i$, so the corresponding schubert variety is of codimension j, where j is the size of the Young diagrams. It is easy to see that this gives a bijection between Young diagrams and Schubert symbols. So the ordering on Schubert symbols induces an ordering on Young diagrams, and it is easy to see that this coincides with the following ordering:

Definition 2.10. For two Young diagrams λ and μ, we define a partial ordering with $\lambda \leq \mu$ if and only if the diagram λ fits into the diagram μ.

Corollary 2.11. *We get* $\mathrm{A}^j(\mathrm{G}(k,n)) = \mathrm{H}^{2j}(\mathrm{G}(k,n)) = \mathbb{Z}^{m_j}$*, where* m_j *is the number of Young diagrams of size* j*, fitting into a rectangle with dimensions* $k \times (n-k)$*. The other cohomology groups vanish.*

Example 2.12. From example 2.9, we see that the even Betti numbers for $\mathrm{G}(2,4)$ are $1, 1, 2, 1, 1$.

3 Flag varieties

Definition 3.1. A *flag* in $\mathbb{C}^n$ of type $(d_1, \ldots, d_k)$ is a chain $\emptyset = \Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \ldots \Sigma_k = \mathbb{C}^n$, where $\dim(\Sigma_i/\Sigma_{i-1}) = d_i$. A flag of type $(1, 1, \ldots, 1)$ is called a *complete flag*. Note that the sum of the d_i's is always equal to n.

Definition 3.2. We define the n-th (complete) *flag variety* as $\mathrm{F}_n = \mathrm{GL}(n)/B$, where $B \leq \mathrm{GL}(n)$ is the subgroup consisting of upper triangular (invertible) matrices.

It is easy to see that the points of $\mathrm{F}(n)$ are exactly the complete flags in $\mathbb{C}^n$. We also have a variety parametrising partial flags:

Definition 3.3. The *partial flag variety* of type $(d_1, \ldots, d_k)$ is defined as

$$\mathrm{F}(d_1, \ldots, d_k) = \mathrm{GL}(n)/P,$$

where $P \leq \mathrm{GL}(n)$ is the lower parabolic subgroup corresponding to $(d_1, \ldots, d_k)$, i.e. the block upper triangular (invertible) matrices with blocks of sizes $d_1, \ldots, d_k$ on the diagonal.

We already looked at the case $\mathrm{G}(k,n) = \mathrm{F}(k, n-k)$. Also note that we indeed have $\mathrm{F}_n = \mathrm{F}(1, \ldots, 1)$.

Lemma 3.4. *Any* $M \in F_n$ *has a unique representant* N *of the following type: there is a permutation* $s \in \mathrm{S}_n$ *such that on row* i*, there is a* 1 *in the* $s(i)$*-th column and such that there are zeroes directly on the right and directly below this* 1*.*

Example 3.5. The representants corresponding to the permutation $3, 2, 5, 1, 4$ are of the form:

$$\begin{pmatrix} * & * & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ * & 0 & 0 & * & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

We can proof lemma 3.4 analogously to lemma 2.2 using linear algebra. Using this lemma, we can apply the same principles as with the Grassmannians: we define Schubert cells and Schubert varieties (indexed by the n-th symmetric group) and show that the Schubert cells give a cell decomposition of the complete flag variety. So the cohomology is again free as an abelian group and we have a combinatorial description of the Betti numbers.

We can use the same methods for the study of partial flag varieties.

References

[Bri05] Michel Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, pages 33–85. Springer, 2005.

[EH10] David Eisenbud and Joe Harris. 3264 & all that: Intersection theory in algebraic geometry. *preparation, to appear*, 2010.

[Ful84] W. Fulton. *Intersection Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer New York, 1984.

[Ful97] William Fulton. *Young tableaux: with applications to representation theory and geometry*, volume 35. Cambridge University Press, 1997.

[Kre07] A Kresch. Flag varieties and schubert calculus. In *Algebraic groups*, pages 73–86, 2007.