Chern classes à la Grothendieck

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Abstract

In this note we introduce Chern classes based on Grothendieck's 1958 paper [4]. His approach is completely formal and he deduces all important properties of Chern classes from a small number of axioms imposed on some given data. These axioms are in particular fulfilled if one inputs the category of smooth quasi-projective varieties with their intersection theory, thus obtaining the familiar theory of Chern classes from this more general setup.

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Conventions We work in the setting of Grothendieck's article, i.e. X will always denote an algebraic variety: an integral, separated scheme of finite type over an algebraically closed field k. If one is so inclined, there is always Fulton's book [1] which treats a more general case.

Disclaimer A big part of this note is shamelessly copied from either Gathmann [3] or Grothendieck [4]. This has not been proofread thoroughly, so if you find any mistakes, please tell me.

1 Projective bundles

Grothendieck's approach is based on taking iterated projective bundles, so in this section we give their definitions and some basic properties. We start out by reviewing some of the theory of vector bundles.

1.1 Vector bundles and locally free sheaves

Definition 1. Let X be a scheme. A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is called locally free of rank r if there is an open cover $\{U_i\}$ of X such that $\mathscr{F}|_{U_i} \cong \mathscr{O}_X^{\oplus r}$ for all i.

Definition 2. A vector bundle of rank r over a field k is a k-scheme F and a k-morphism $\pi : F \to X$, together with an open covering $\{U_i\}$ of X and isomorphisms

(1) $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}_k^r,$

such that the automorphism $\psi_i \circ \psi_j^{-1}$ of $(U_i \cap U_j) \times \mathbb{A}^r$ is linear in the coordinates of $\mathbb{A}^r$.

Proposition 3. There is a one-to-one correspondence between vector bundles of rank r on X and locally free sheaves of rank r on X.

Proof. To a vector bundle F one associates the sheaf $\mathscr{F}$ defined by

(2) $\mathscr{F}(U) = \{k - \text{morphisms } s : U \to F \text{such that } \pi \circ s = \mathrm{id}_U\},$

which is called the sheaf of sections. Conversely, let $\mathscr{F}$ be a locally free sheaf. Take an open cover $\{U_i\}$ of X such that there are isomorphisms $\psi_i : \mathscr{F}|_{U_i} \to \mathscr{O}_{U_i}^{\oplus r}$. Now glue the schemes $U_i \times \mathbb{A}_k^r$ together along the isomorphism

(3) $(U_i \cap U_j) \times \mathbb{A}_k^r \to (U_i \cap U_j) \times \mathbb{A}_k^r : (p, x) \mapsto (p, (\psi_i \circ \psi_j^{-1})(x)).$

Notice that linearity follows from the fact that $\psi_i \circ \psi_j^{-1}$ is a morphism of $\mathscr{O}_X$-modules. □

The following lemmas are easy to prove and show that locally free sheaves on an arbitrary scheme are 'nice', i.e. all linear algebra constructions go through. The second lemma is used very, very often in the literature but is easily forgotten, at least by the author.

Lemma 4. Locally free sheaves are closed under direct sums, tensor products, symmetric products, exterior products, duals and pullbacks.

Lemma 5. Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of locally free sheaves of ranks f, g and h on a scheme X. Then $\wedge^g \mathscr{G} \cong \wedge^f \mathscr{F} \otimes \wedge^h \mathscr{H}$.

From now on we will use the terms 'vector bundle' and 'locally free sheaf' interchangeably.

1.2 Projective bundles

Informally, for a given vector bundle F on X, the associated projective bundle $\mathbb{P}(F)$ replaces each fibre $F_x, x \in X$ with its projectivization $\mathbb{P}(F_x)$, so $\mathbb{P}(F)_x = \mathbb{P}(F_x)$. Let us make this precise.

Definition 6. Let $\pi : F \to X$ be a vector bundle of rank r on a scheme X. The projective bundle $\mathbb{P}(F)$ is defined by glueing $U_i \times \mathbb{P}^{r-1}$ to $U_j \times \mathbb{P}^{r-1}$ along the isomorphisms

(4) $\quad (U_i \cap U_j) \times \mathbb{P}^{r-1} \to (U_i \cap U_j) \times \mathbb{P}^{r-1} : (p, x) \mapsto (p, \psi_{i,j} x).$

One says that $\mathbb{P}(F)$ is a projective bundle of rank $r - 1$ on X.

Notice that the corresponding projection morphism $\mathrm{pr} : \mathbb{P}(F) \to X$ is proper (since properness if 'local on the base'), which is not the case for vector bundles.

We should remark that the general construction of (projective) bundles is as follows. Starting from a locally free sheaf $\mathscr{F}$, the associated vector bundle (repsectively projective bundle) is defined to be

(5) $\quad$ **Spec** $S(\mathscr{F})$, respectively **Proj** $S(\mathscr{F})$,

i.e. one takes the relative spec (proj) of the symmetric algebra associated to $\mathscr{F}$, see Hartshorne, Ex. II.5.18, II.7.10. It is important to note that if X is a variety, then **Proj** $S(\mathscr{F})$ is as well. The property of being (quasi-)projective also passes to projective bundles. This follows from a more general result on blowups, see Hartshorne, Prop. II.7.16. This is not true for ordinary vector bundles **Spec** $S(\mathscr{F})$!

We now want to construct a canonical line bundle on $\mathbb{P}(F)$, called the tautological subbundle. Let $\pi : \mathbb{P}(F) \to X$ denote the projection map, and consider the pullback bundle $\pi^* F$ on $\mathbb{P}(F)$. The corresponding open covering of $\mathbb{P}(F)$ is $(U_i \times \mathbb{P}^{r-1})$ and the bundle is made up of patches $(U_i \times \mathbb{P}^{r-1}) \times \mathbb{A}^r$, which are glued along the isomorphisms

(6) $\quad (U_i \cap U_j) \times \mathbb{P}^{r-1}) \times \mathbb{A}^r \to (U_i \cap U_j) \times \mathbb{P}^{r-1}) \times \mathbb{A}^r : (p, x, y) \mapsto (p, \psi_{i,j} x, \psi_{i,j} y).$

Definition 7. The tautological subbundle $L_F^\vee$ on $\mathbb{P}(F)$ is the rank 1 subbundle of $\pi^* F$ given locally by equations

(7) $\quad x_i y_j = x_j y_i,$

for $i, j = 1, \ldots, r$.

Geometrically, the fiber of $L_F^\vee$ over a point $(p, x) \in \mathbb{P}(F)$ is the line in the fiber F_p whose projectivization is x. The reason for taking duals will become clear later on.

1.3 The splitting principle

One would like to have a nice 'composition' series for any vector bundle F of rank r over X. By this we mean a filtration by subbundles

(8) $\quad 0 = F_0 \subset F_1 \subset \ldots \subset F_{r-1} \subset F_r = F,$

such that F_i/F_{i-1} is a line bundle on X. In general this is not possible, but it is possible after pulling back the bundle to another variety. This theorem is often referred to as the 'splitting principle'.

Theorem 8. Let F be a rank r vector bundle on X, then there exists a variety Y and a morphism $f : Y \to X$ such that f^*F has a filtration by vector bundles

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{r-1} \subset F_r = f^*F, \tag{9}$$

such that $\operatorname{rk} F_i = i$. In fact, Y can be constructed as an iterated projective bundle.

Proof. We prove the statement by induction. For $\operatorname{rk} F = 1$, it is trivial, so suppose F has rank strictly greater than 1. Let $Y' := \mathbb{P}(F^\vee)$ with associated morphism $f' : Y' \to X$. Remember that $L^\vee_{F^\vee} \subset (f')^*F^\vee$ denotes the tautological line bundle. By dualizing we get a surjective morphism such that the kernel is a vector bundle:

$$0 \to \tilde{F} \to (f')^*F \to L_{F^\vee} \to 0. \tag{10}$$

Now $\tilde{F}$ has rank one less than the rank of F so by induction there is a variety Y and a morphism $f'' : Y \to Y'$ such that $(f'')^*\tilde{F}$ has a filtration with subquotient line bundles. It now suffices to set $f = f' \circ f''$ such that f^*F has a filtration

$$0 = F_0 \subset F_1 \subset \ldots \subset F_{r-1} = (f'')^*\tilde{F} \subset f^*F \tag{11}$$

and we're done. From the proof it is clear that Y can be constructed as iterated projective bundle. □

2 Input and axioms

In this section we introduce the formal data Grothendieck needs to obtain a nice theory of Chern classes. Chern classes are invariants associated to a vector bundle E over a variety X. The idea, which Grothendieck attributes to Chern, is to use the multiplicative structure on the ring of classes of algebraic cycles on the projective bundle associated to a vector bundle E on X to obtain an explicit construction of the Chern classes associated to E. In the setting we are interested in, the Chern classes live in the Chow ring $CH^\bullet(X)$ of X, but in fact, this is not the only possibility, and Grothendieck's framework is general enough to cover other interesting case, for which we refer to the article.

2.1 Input

Let $\mathcal{V}$ denote some category of smooth algebraic varieties over k, where the morphisms are morphisms of algebraic varieties.

This category has to satisfy:

$$V1 : \text{If } X \in \mathcal{V}, \text{ and } F \text{ is a vector bundle on } X, \text{ then } \mathbb{P}(F) \in \mathcal{V}.$$

Further, one needs the following as input:

1. A contravariant functor $A : \mathcal{V} \to \mathrm{GCR}$, where GCR denotes the category of graded commutative unital rings, i.e. one has $xy = (-1)^{\deg x \deg y} yx$;
2. A functorial homomorphism of abelian groups $p_X : \mathrm{Pic}(X) \to A^2(X)$, for $X \in \mathcal{V}$;
3. Let $i : Y \hookrightarrow X \in \mathcal{V}$ be a closed algebraic subvariety of constant codimension p in X, such that Y is also in $\mathcal{V}$. Then there is a group homomorphism

   (12) $i_* : A(Y) \to A(X)$,

   increasing the degree by $2p$.

We need some more notation: for a morphism $f : X \to Z$ in $\mathcal{V}$, we will denote $f^* := A(f) : A(Z) \to A(X)$. The unit of $A(X)$ will be denoted 1_X, and for i and Y as in 3. above, we define

(13) $p_X(Y) := i_*(1_Y)$.

Also, if F is a vector bundle on X, remember that $L_F^\vee$ denotes the tautological subbundle of $\mathbb{P}(F)$. Then using 2. above, we define ξ_F as follows

(14) $\xi_F := p_{\mathbb{P}(F)}(L_F) \in A^2(\mathbb{P}(F))$.

Also notice that $A(\mathbb{P}(F))$ can be considered as a left $A(X)$-module by applying the functor A to the projection morphism $\mathrm{pr} : \mathbb{P}(F) \to X$. We will refer to the input items as $I.1, I.2, I.3$.

Smooth projective varieties We briefly review this input for $\mathcal{V}$ be the category of smooth projective varieties over k. Details can be found in [1, 2]. The condition $V1$ is satisfied by the remark in the section on projective bundles. The functor $A = CH$, which sends a variety to its Chow ring with doubled degree, i.e. we put CH^i in degree $2i$, since the Chow ring is commutative (for other natural occurring A this is not the case. For a morphism of nonsingular varieties $f : X \to Y$, one can define a pullback by $\alpha \mapsto \gamma_f^*(\alpha \times [Y])$, where γ_f denotes the graph of f. That this works and is well defined can be found in [2]. The functorial morphism p_X is just the map sending a Cartier divisor to its associated Weil divisor: a Cartier divisor is represented by the data $\{(U_i), f_i\}$, and one sends this to the Weil divisor $\sum_V \mathrm{ord}_V \cap U_i(f)[V]$, where V runs through the codimension 1 subvarieties of X. Notice that this is in fact an isomorphism for all smooth n-dimensional schemes, which uses a deep theorem from commutative algebra, the Auslander-Buchsbaum theorem, which says that regular local rings are unique factorization domains. Finally, for the third datum the induced pushforward map just sends the class of a subvariety $[Z]$ of codimension l in Y to $[Z]$. So in X it is of codimension $p + l$, corresponding to a degree increase of $2p$.

2.2 Axioms

Given the input in the previous section, Grothendieck requires them to satisfy the following four axioms.

1. For $X \in \mathcal{V}$, and F a vector bundle of rank r on X, the elements

   (15) $1_{\mathbb{P}(F)}, \xi_F, \xi_F^2, \ldots, \xi_F^{r-1}$

   form a basis of the $A(X)$-module $A(\mathbb{P}(F))$.
2. For $X \in \mathcal{V}$, L a line bundle on X, and s a regular section of L transversal to the zero section, such that $s^{-1}(0) \in \mathcal{V}$, one has

   (16) $p_X(s^{-1}(0)) = p_X(L)$.
3. For $Z \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} X$, all belonging to $\mathcal{V}$, one has

   (17) $(j \circ i)_* = j_* \circ i_*$.
4. For $Y \overset{i}{\hookrightarrow} X$, both belonging to $\mathcal{V}$, one has

   (18) $i_*(bi^*(a)) = i_*(b)a$,

   for $a \in A(X), b \in A(Y)$.

We will refer to these as $A.1$ through $A.4$.

Smooth projective varieties For smooth projective varieties, axiom 1 is quite hard, and can be found for example as Theorem 3.3 (b)in [1]. Axiom 2 follows immediately from the familiar correspondence between Cartier divisors and line bundles. Axiom 3 is immediate from the definition of the pushforward we gave in the Inputs section. Axiom 4 is known as the 'projection' formula and can also be found in [1].

2.3 Some fundamental lemmas

The input and axioms can be split in two subsets, each with a specific purpose. The map i_* discussed in $I.3$, and the axioms pertaining to i_*, namely $A.2$,$A.3$ and $A.4$ have as main purpose the proof of the following technical lemma.

Lemma 9. Let $X \in \mathcal{V}$, F a vector bundle of rank r on X, and s a regular section of F. Further, let

(19) $F = F_0 \supset F_1 \supset \cdots \supset F_{r-1} \supset F_r = 0$

be a decreasing sequence of subbundles of F, such that $\mathrm{rk}(F_i) = r - i$. For each $i = 1, \ldots, r$, define

(20) $Y_i = \{x \in X \mid s(x) \in F_i\}$,

and suppose that Y_i is a non-singular subvariety of X, which is contained in $\mathcal{V}$. Now let s_i be the section of $(F_i/F_{i+1})|_{Y_i}$ induced by s, and suppose that all the s_i are transversal to the zero section. If one finally defines

(21) $\xi_i = p_X(F_{i-1}/F_i)$,

then one concludes that

(22) $p_X(Y_r) = \prod_{1 \le i \le r} \xi_i$.

Proof. We will prove that

$$(23)\quad p_X(Y_j) = \prod_{1\le i\le j} \xi_j,$$

by induction on j. For $j = 1$ this is just axiom 2. Denote by $Y_{j+1} \overset{i}{\hookrightarrow} Y_j \overset{u_j}{\hookrightarrow} X$ the respective inclusion. Now one finds

$$(24)\quad \begin{aligned} p_{Y_j}(Y_{j+1}) &=^{A.2} p_{Y_j}([F_j/F_j+1]|_{Y_j}) \\ &= p_{Y_j}(u_j^*[F_j/F_{j+1}]) \\ &=^{I.2} u_j^* p_X(F_j/F_{j+1}) \\ &= u_j^*(\xi_{j+1}). \end{aligned}$$

Now apply u_{j*} to the equality and in the next equality $A.3$ on the LHS to get

$$(25)\quad \begin{aligned} u_{j*}i_*(1_{Y_{j+1}}) &= u_{j*}u_j^*\xi_{j+1} \\ p_X(Y_{j=1}) &= u_{j*}(1_{Y_j}\cdot u_j^*\xi_{j+1}) \\ &=^{A.4} u_{j*}(1_{Y_j})\cdot \xi_{j+1} \\ &= p_X(Y_j)\xi_{j+1}. \end{aligned}$$

Now it suffices to use the induction hypothesis. □

Perhaps more important still is the following corollary which immediately follows by combining the lemma with $A.2$.

Corollary 10. Under the conditions of lemma 9, and assuming that s vanishes nowhere, one has

$$(26)\quad \prod_{1\le i\le r} \xi_i = 0.$$

Then there is one more lemma, which is crucial for the uniqueness property of Chern classes.

Lemma 11. Let $X \in \mathcal{V}$, and F a vector bundle of rank r on X. Let $f : Y \to X$ be the morphism obtained by the splitting principle. Then the induced morphism

$$(27)\quad f^* : A(X) \to A(Y)$$

is injective.

Proof. By our construction of the Y as iterated projective bundle, we can assume to be working with $\mathbb{P}(F)$, and here the statement follows immediately from $V1$ and axiom 1, since it says $\xi_F^0 = 1_{\mathbb{P}(F)}$ is free over $A(X)$. An inductive argument then shows the claim. □

3 Chern classes

To introduce Chern classes and describe their characterising properties, we will need corollary 10, $I.1, I.2$ and $A.1$.

From axiom 1, we immediately find that there exist unique elements $c_i(F) \in A^{2i}(X)$ for every natural number $i \geq 0$ such that

$$(28) \quad \sum_{i=0}^{r} c_i(F)(\xi_F)^{r-i} = 0 \qquad c_0(F) = 1 \qquad c_i(F) = 0 \text{ for } i > r.$$

Definition 12. The $c_i(F)$ defined above is called the i-th Chern class of F. The sum of all Chern classes is denoted

$$(29) \quad c(F) = \sum_i c_i(F),$$

and is called the total Chern class of F.

The following theorem completely describes the Chern classes in terms of their properties, which are tailored to computability.

Theorem 13. The Chern classes defined above satisfy the following 3 properties:

1. Functoriality: let $f : X \to Y$ be a morphism in $\mathcal{V}$, and F a vector bundle on Y, then

   $$(30) \quad c(f^*F) = f^*(c(E));$$

2. Normalization: let L be a line bundle on $X \in \mathcal{V}$. Then

   $$(31) \quad c(L) = 1 + p_X(L);$$

3. Additivity: for $X \in \mathcal{V}$, and $0 \to F' \to F \to F'' \to 0$ an exact sequence of vector bundles on X, one has

   $$(32) \quad c(F) = c(F')c(F'').$$

Moreover, these 3 properties uniquely characterize Chern classes given the input and axioms.

Let us first show the uniqueness statement, since this basically tells one how to actually compute a Chern class from the input. Let $f : Y \to X$ be the map from the splitting principle. Since Y is an iterated projective bundle, by $V1$ we know that $Y \in \mathcal{V}$. By lemma 11, we know the associated map $f^* : A(X) \hookrightarrow A(Y)$ is injective. So if we know $f^*(c(F))$, then we know $c(F)$. But now by functoriality, one has $f^*(c(F)) = c(f^*F)$. We know from the splitting principle that f^*F has a filtration by line bundles, so from additivity one obtains

$$(33) \quad c(F) = \prod_{i=1}^{r} c(F_{i-1}/F_i) = \prod_{i=1}^{r} (1 + p_X(F_{i-1}/F_i)),$$

where we used normalization in the last step.

Functoriality is not that hard, but we skip the proof. Let us first show normallity. Since L is a line bundle, we have $\mathbb{P}(L) \cong X$ and $L_L^\vee$, the tautological line bundle, is just L. So now

$$(34) \quad \xi_L = p_{\mathbb{P}(L)}(L_L) = p_X(L^\vee) = -p_X(L).$$

Now writing out equation 28 for a line bundle we get $\xi_L + c_1(L) = 0$, so $c_1(L) = p_X(L)$ and normality follows since $c_0(L) = 1$.

It is only additivity that requires real work. Let us give a sketch of a proof. Using functoriality and the splitting principle in a similar way as before, one can reduce to the question whether additivity holds when F' and F'' have a complete filtration by subbundles. Then obviously F also has such a filtration and it will suffice to show that for every composition series of F', F'' and F thus obtained, the equation 33 holds. So in effect, it remains to show equation 33 for a vector bundle F that has a complete filtration by subbundles. Consider the following diagram

$$\begin{tikzcd} f^*F \arrow[d] & F \arrow[d] \\ \mathbb{P}(F) \arrow[r, "f"] & X \end{tikzcd} \tag{35}$$

and let $L_F^\vee$ denote the tautological subbundle of $\mathbb{P}(F)$. Let $(F_i)_i$ be a complete filtration by subbundles of F. Defining $F' = L_F \otimes f^*F$, the filtration on F gives a filtration on this bundle with factors $F'_{i-1}/F'_i = L_F \otimes (f^*F_{i-1}/f^*F_i)$, so in the Picard group of $\mathbb{P}(F)$, we get the equality

$$F'_{i-1}/F'_i = L_F + (f^*F_{i-1}/f^*F_i). \tag{36}$$

Applying the group morphism $p_{\mathbb{P}(F)}$, we get

$$p_{\mathbb{P}(F)}(F'_{i-1}/F'_i) = \xi_F + \xi'_i, \tag{37}$$

where $\xi'_i = p_{\mathbb{P}(F)}(f^*F_{i-1}/f^*F_i)$, just like in lemma 9. Now from the inclusion $L_F^\vee \hookrightarrow f^*F$, one obtains a non-vanishing section s of F' that turns out to be transversal to the zero section (I'm skipping the transversality proof). This allows one to apply corollary 10 to obtain

$$\prod_{1 \le i \le r} (\xi_F + \xi'_i) = 0. \tag{38}$$

This says that the $c_i(F)$, defined by equation 28, are elementary symmetric functions in the ξ'_i, which is exactly what equation 33 says (remember that this equation involves a pullback).

Using the theory of symmetric functions, one can deduce formula for the Chern classes of tensor products, exterior products and duals.

4 Appendix 1: Example of a projective bundle

Let $X = \mathbb{P}^1$ and let F be the rank 2 vector bundle $\mathcal{O}_X \oplus \mathcal{O}_X(-1)$ on X. Then $\mathbb{P}(F)$ is a projective bundle of rank 1, so it is a scheme of dimension 2. Our claim is that $\mathbb{P}(F)$ is isomorphic to the blow-up of $\mathbb{P}^2$ in a point p.

At least intuitively, it is clear that the blow-up should be a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, since one can project onto the exceptional divisor. So we are looking for a rank 2 bundle F

on the projective line. Since every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles, we know F is of the form $\mathcal{O}(d') \oplus \mathcal{O}(d')$. Now tensoring a vector bundle with a line bundle multiplies the transition functions by a scalar, so it does not change the associated projective bundle, so we can assume that $F = \mathcal{O} \oplus \mathcal{O}(-n)$, for some $n \geq 0$. It is not too hard to see that this $-n$ gives rise to a curve of self-intersection $-n$ so in fact $n = 1$.

Let us check this formally: $\mathbb{P}(F)$ is obtained by glueing two copies U_1 and U_2 of $\mathbb{A}^1 \times \mathbb{P}^1$ along the isomorphism

$$(\mathbb{A}^1\backslash\{0\}) \times \mathbb{P}^1 \to (\mathbb{A}^1\backslash\{0\}) \times \mathbb{P}^1 : (q, (x_1 : x_2)) \mapsto (1/q, (x_1 : qx_2)). \tag{39}$$

The changes in the affine coordinate q correspond to the glueing one uses to obtain $\mathbb{P}^1$. The first projective coordinate gets sent to itself, because of $\mathcal{O}_X$ and the second one to qx_2 because of $\mathcal{O}_X(-1)$.

The blow-up of $\mathbb{P}^2$ in $p = (1 : 0 : 0)$ on the other hand is given by

$$\mathrm{Bl}_p(\mathbb{P}^2) = \{((x_0 : x_1 : x_2), (y_1 : y_2)) \mid x_1 y_2 = x_2 y_1\} \subset \mathbb{P}^2 \times \mathbb{P}^1. \tag{40}$$

An explicit isomorphism between the two varieties is given by

$$\begin{aligned} U_1 &\cong \mathbb{A}^1 \times \mathbb{P}^1 \to \mathrm{Bl}_p(\mathbb{P}^2) : (q, (x_1, x_2)) \mapsto ((x_1 : qx_2 : x_2), (q : 1)), \\ U_2 &\cong \mathbb{A}^1 \times \mathbb{P}^1 \to \mathrm{Bl}_p(\mathbb{P}^2) : (q, (x_1 : x_2)) \mapsto ((x_1 : x_2 : qx_2), (1 : q)), \end{aligned} \tag{41}$$

and note that these morphisms are compatible with the glueing isomorphism (39).

Now let us compute the Chow groups of $\mathbb{P}(F)$. The computation is based on the following proposition.

Proposition 14. Let X be a scheme stratified by affine spaces, i.e. there is a filtration by closed subschemes

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X, \tag{42}$$

such that $X_k \backslash X_{k-1} = \mathbb{A}^k \amalg \cdots \amalg \mathbb{A}^k$, with $\mathbb{A}^k$ appearing a_k times. Then $A_k(X) \cong \mathbb{Z}^{a_k}$.

The projective plane has a stratification $\mathbb{A}^2 \amalg \mathbb{A}^1 \amalg \mathbb{A}^0$, so identifying $\mathbb{A}^0$ with p, the projective bundle $\mathbb{P}(F)$ has a stratification $\mathbb{A}^2 \amalg \mathbb{A}^1 \amalg \mathbb{A}^1 \amalg \mathbb{A}^0$. Denoting by q any point in $\mathbb{P}(F)$, by L the strict transform of a line in $\mathbb{P}^2$ through p and by E the exceptional divisor, one finds

$$\begin{aligned} A_0(X) &= \mathbb{Z}[q] \\ A_1(X) &= \mathbb{Z}[L] \oplus \mathbb{Z}[E] \\ A_2(X) &= \mathbb{Z}[\mathbb{P}(F)]. \end{aligned} \tag{43}$$

In fact, one can explicitly check that there is no relation in $A_1(X)$. Suppose that $n[L]+m[E] = 0$. Let $\pi : \mathbb{P}(F) \to \mathbb{P}^2$ be the projection to the base of the blow-up. This is proper and

$$0 = \pi_*(0) = \pi_*(n[L] + m[E]) = n[M] + m \cdot 0 \in A_1(\mathbb{P}^2), \tag{44}$$

where $[M]$ is the class of a line in $\mathbb{P}^2$, so $n = 0$. Denote by $f : \mathbb{P}(F) \to \mathbb{P}^1$ the $\mathbb{P}^1$-bundle map. Then

$$0 = f_*(0) = f_*(n[L] + m[E]) = n \cdot 0 + m[\mathbb{P}^1], \tag{45}$$

so $m = 0$. One can also show that for any line H in $\mathbb{P}(F)$ not intersecting E, one has $[H] = [L] + [E] \in A_1(\mathbb{P}(F))$.

Now $\mathrm{Pic}(\mathbb{P}(F)) = \mathbb{Z}[H] \oplus \mathbb{Z}[E]$, so to compute the intersection products it will thus suffice to compute $H^2, H \cdot E$ and E^2. First of all, clearly $H^2 = 1$ and $H \cdot E = 0$. Now

(46) $E^2 = E \cdot (H - L) = E \cdot H - E \cdot L = 0 - 1 = -1.$

All in all, the Chow ring has the following presentation

(47) $A^\bullet(\mathbb{P}(F)) \cong \mathbb{Z}[x, y, z]/(x^2 = z, xy = 0, y^2 = -z),$

where $\deg(x) = \deg(y) = 1$ and $\deg(z) = 2$.

5 Appendix 2: Example of a Chern class computation

Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$. It is not hard to check that

(48) $\mathrm{CH}^\bullet(X) \cong \mathbb{Z}[x, y]/(x^2, y^2)$

and intuitively, this is clear since lines in the same ruling do not intersect and lines in a different ruling intersect in one point, corresponding to the polynomial xy. We consider the problem of determining the kernel of the morphism

(49) $\mathrm{Hom}(\mathcal{O}(0,1), \mathcal{O}(1,1)) \otimes \mathcal{O}(0,1) \to \mathcal{O}(1,1) \to 0,$

which should be a vector bundle F of rank 1. There is very easy way of doing this, pointed out to me by Dennis Presotto, using the (told you I always forget) lemma 5, but at least the following illustrates how a Chern class computation can be carried out. We'll do it step by step, to clearly illustrate what is going on.

By additivity, we know that

(50) $c(F) \cdot c(\mathcal{O}(1,1)) = c(\mathcal{O}(0,1) \oplus \mathcal{O}(0,1)).$

Now using additivity and then normality, the RHS becomes

(51) $c(\mathcal{O}(0,1) \oplus \mathcal{O}(0,1)) = c(\mathcal{O}(0,1)) \cdot c(\mathcal{O}(0,1)) = (1 + y)(1 + y) = 1 + 2y.$

For the LHS we find using normality and the group operation in $\mathrm{Pic}(X)$ that

(52) $c(\mathcal{O}(1,1)) = 1 + p_X(\mathcal{O}(1,1)) = 1 + p_X(\mathcal{O}(1,0)) + p_X(\mathcal{O}(0,1)) = 1 + x + y.$

Now it is easy to check that

(53) $c(F) = 1 - x + y,$

and using the isomorphism between $\mathrm{CH}^1(X)$ and $\mathrm{Pic}(X)$, we see that $F = \mathcal{O}(-1,1)$.

References

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