Growth of Algebras and the Gelfand-Kirillov Dimension

Dennis Presotto

February 10, 2014

In the non-commutative case the notion of Krull dimension does not work as well anymore because of a lack of symmetry between left and right ideals. Therefor another way of saying that an algebra should be small is stating that it should have finite GK-dimension or equivalently: it should have polynomial growth. Below we briefly explain what this means. Our main reference is [2].

We first define growth of an algebra A that is finitely generated over a field k. This notion is similar to growth of finitely generated groups as considered in [1] and when we mention growth of groups we will use the same notation as used in loc. cit.

Definition 0.1. Let k be a field and A a finitely generated k-algebra. Suppose that V is a finite dimensional generating subspace for A, i.e.

$$A = \bigcup_{n \in \mathbb{N}} V_n \text{ where } V_n = k + V + V^2 + \ldots + V^n$$

We associate a growthfunction to V as follows:

$$d_V : n \mapsto dim(V_n)$$

There are some remarks concerning this definition.

Remark.

1. One can notice that such a finite dimensional generating subspace always exists, since if A is finitely generated by elements $a_1, \ldots, a_m$, we may set
   $V = ka_1 + \ldots + ka_m$.
2. With any choice of V satisfying $1_A \in V$ we have $V^i \subset V^{i+1}$ and thus $d_V(n) = dim(V^n)$ is an increasing function. It is clear that if A is finite dimensional this function will become constant for large values of n.

3. In case A is a graded algebra which is finitely generated in degree 1, then the growth function is closely related the Hilbert function: $n \mapsto dim_k(A_n)$. For one can take $V = ka_1 + \ldots + ka_m$ where the a_i are the degree 1 generators, in which case $A_n = V^n$. Hence if the Hilbert function is a polynomial of degree λ, then d_V is a polynomial of degree $\lambda + 1$.

4. The definition of d_V depends on the choice of generating subspace. This dependence may be removed by introducing a suitable equivalence relation.

Definition 0.2. Let Φ denote the set of all functions $f : \mathbb{N} \to \mathbb{R}^+$ that are eventually monotone increasing, i.e.

$$f(n+1) \geq f(n) \quad \text{for almost all } n \in \mathbb{N}.$$

For $f, g \in \Phi$ we say that f is dominated by g, written $f \preccurlyeq g$, if and only if there exist positive integers c and m such that

$$f(n) \leq cg(mn) \quad \text{for almost all } n \in \mathbb{N}.$$

We say that f and g are equivalent if and only if $f \preccurlyeq g$ and $f \succcurlyeq g$. This will be denoted by $f \sim g$. The "growth" of $f \in \Phi$ is the equivalence class $\mathcal{G}(f) \in Phi/$ of f. The relation $\preccurlyeq$ induces an order $\leq$ on $\Phi/\sim$.

If we consider $f(n) = n + 1$ and $g(n) = n^2$ then $f \preccurlyeq g$ but the condition $f(n) \leq cg(mn)$ is always false for $n = 0$. This shows that the restriction to "almost all n" in the definition of $\preccurlyeq$ is important.
The following results follow immediatly from the definition:

- For any real number γ we denote $p_\gamma : n \mapsto n^\gamma$ and $P_\gamma = \mathcal{G}(p_\gamma)$. If $f(n) = p_a$ and $g(n) = p_b$ then $f \preccurlyeq g \Leftrightarrow a \leq b$. As a result two polynomial functions are equivalent iff they have the same degree.

- Any pair of exponential functions with bases > 1 are equivalent, i.e. if $f(n) = a^n$ and $g(n) = b^n$ with $a, b > 1$ then $\mathcal{G}(f) = \mathcal{G}(g)$. We will denote their growth by ε_1. More generally $\varepsilon_\epsilon = \mathcal{G}\left(n \mapsto e^{n^\epsilon}\right)$ and obviously $\varepsilon_\epsilon \leq \varepsilon_\eta \Leftrightarrow \epsilon \leq \eta$.

The next proposition states that modulo this equivalence the growth functions do not depend on the generating set.

Proposition 0.3. *Let A be a finitely generated k-algebra with finite dimensional generating subspaces V and W, then $\mathcal{G}(d_V) = \mathcal{G}(d_W)$.*

Proof. By symmetry we only have to show $d_V \preccurlyeq d_W$.
Since $A = \bigcup_{n \in \mathbb{N}} k + V + V^2 + \ldots + V^n$ there exists a $c \geq 1$ such that $V \subset 1 + W + \ldots + W^c$, hence $d_V(n) \leq d_W(cn)$ □

As a result we can say things like the growth of an algebra by simply choosing any generating subspace V. Since the choice of this subspace is arbitrary we will always assume $1_A \in V$ for simplicity.

Definition 0.4. With the same notations as above we set $\mathcal{G}(A) = \mathcal{G}(d_V)$ to be the growth of the algebra A and A is said to have

- polynomial growth if $\mathcal{G}(A) \leq P_m$ for some $m \in \mathbb{N}$
- exponential growth if $\mathcal{G}(A) = \varepsilon_1$

This terminology is very similar to the one used for groups. In fact one has the following result

Theorem 0.5. *Let G be a finitely generated group.*
Then $k[G] = \{$finite sums $\sum_{g\in G} x_g \cdot g\}$ is a finitely generated k-algebra having the same (class of) growth as G.

Proof. Since growth of a group does not depend on the generating subset it's sufficient to show that $\gamma_G^S \sim d_V$ for some finite generating subset S for G and finite dimensional generating subspace V for $k[G]$.
Take S such that it contains all inverses of it's elements as well as the identity element, then $k[S]$ is a finite dimensional generating subspace of $k[G]$ containing $1_{k[G]}$. Moreover this choice implies $\gamma_G^S = d_{k[S]}$ □

Examples.

- Consider $G = \mathbb{Z}^m$, it was found in [1] that G has polynomial growth (it grows like P_m). So $k[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}] \cong k[\mathbb{Z}^m]$ is a finitely generated k-algebra having polynomial growth
- On the contrary any free algebra $k\langle x_1, \ldots, x_m\rangle$ has exponential growth if $m \geq 2$.

  For this take $V = kx_1 + \ldots kx_m$, then $d_V(n) = \displaystyle\sum_{i=0}^{n} m^i = \frac{m^{n+1}-1}{m-1}$.

1 Gelfand-Kirillov dimension

Now we want to introduce the Gelfand-Kirillov dimension. But before we do so, we first give an arithmatic result.

Proposition 1.1. *Let* $f, g \in \Phi$ *and denote* $\log_n(f(n)) = \dfrac{\log(f(n)))}{\log(n)}$. *Then*

1. *The following 3 are equal:*

$$\begin{aligned} \overline{\lim_{n\to\infty}} \log_n f(n) &= \inf\{\lambda \in \mathbb{R} \mid f(n) \leq n^{\lambda} \textit{ for almost all } n\} \\ &= \inf\{\lambda \in \mathbb{R} \mid \mathcal{G}(f) \leq P_{\lambda}\} \end{aligned}$$

   where we set $\inf(\emptyset) = \infty$.

2. *If* $\mathcal{G}(f) = \mathcal{G}(g)$ *then* $\overline{\lim_{n\to\infty}} \log_n(f(n)) = \overline{\lim_{n\to\infty}} \log_n(g(n))$.

Proof. 1. (Similar to the proof of [2, Lemma 2.1])
Denote the three numbers mentioned by r, s and t (in that order).
It's easy to see that if one of them is infinite, so are the others. It's also immediate that $t \leq s$.

To prove $s \leq r$ one notices that for any $\epsilon > 0$ fixed we have $f(n) \leq n^{r+\epsilon}$ for almost all n.

We are only left to show $r \leq t$, this can be done by contradiction.
So suppose $r > t$. Take $\epsilon = \frac{r-t}{3} > 0$. Since $\mathcal{G}(f) \leq P_{t+\epsilon}$ there are positive integers c, m such that $f(n) \leq c \cdot (mn)^{t+\epsilon} \leq n^{t+2\epsilon} = n^{r-\epsilon}$ for almost all n. This is a contradiction since it would imply $\overline{\lim} \log_n f(n) \leq r - \epsilon$.

2. Trivial by 1.

$\square$

Remark that 2. of the above proposition implies that there is a well defined function: $\overline{\lim_{n\to\infty}} \log_n : \Phi/\sim \to \mathbb{R}^+$. It also makes sure the following is well defined

Definition 1.2. The Gelfand-Kirillov dimension of a finitely generated k-algebra A is defined as

$$GKdim(A) = \overline{\lim_{n\to\infty}} \log_n(d_V(n))$$

where V is an arbitrary finite dimensional generating subspace.
In case A is not finitely generated, one defines:

$$GKdim(A) = \sup\{\ GKdim(\tilde{A}) \mid \tilde{A} \subset A \text{ is a finitely generated sub-}k\text{-algebra }\}$$

As most algebras we encounter will be finitely generated, we are mainly interested in the "easy" part of the definition.

An immediate corollary of Propostion 1.1 is:

Corollary 1.3. *Let A be a finitely generated k-algebra,*

- *If $\mathcal{G}(A) = P_\lambda$ then $GKdim(A) = \lambda$*
- *If $GKdim(A) < \infty$ then A has polynomial growth.*
- *If $\mathcal{G}(A) = \varepsilon_1$ then $GKdim(A) = \infty$*

Some caution is required concerning the definition of polynomial growth as was introduced in Definition 0.4. For $\overline{\lim\limits_{n\to\infty}} \log_n f(n) = \lambda$ does not imply $\mathcal{G}(f) = P_\lambda$.
$f(n) = \log(n+1) \cdot n^\lambda$ serves as a counter example.
Remark however that $\log(n+1) \cdot n^\lambda \preccurlyeq n^{\lambda+1}$ hence it does have polynomial growth.

Examples.

- If A is a finitely generated k-algebra, $GKdim(A) = 0$ if and only if A is finite dimensional so the Gelfand Kirillov dimension somehow measures how much a finitely generated algebra fails to be finite dimensional[1]
- $GKdim(k[x_1, \ldots, x_m]) \leq m$. To see this take $V = kx_1 + \ldots kx_m$ then $V_n = 1 \oplus V \ldots \oplus V^n$, hence

$$d_V(n) = \sum_{i=0}^{n} dim(V^n) = \sum_{i=0}^{n} \binom{m+i-1}{m-1} \leq (n+1) \cdot (m+n)^{m-1} \preccurlyeq n^m$$

  (In fact $GKdim(k[x_1, \ldots, x_m]) = m$ as will follow from Proposition 2.2 or Corollary 2.3)
- Any free algebra $k\langle x_1, \ldots, x_m \rangle$ with $m \geq 2$ has infinite $GK\!-\!dimension$. (This follows from the second example after Theorem 0.5)

We now give some propositions without proofs:

Proposition 1.4. *Let A be any k-algebra If A is a finitely generated commutative ring, then the Gelfand Kirillov dimension equals its Krull dimension (and hence is a nonnegative integer).*

Proof. [2, Theorem 4.5] □

[1]In the more general case $GKdim(A) = 0$ if and only if A is locally finite dimensional. I.e. every finitely generated sub-k-algebra is finitely generated

Proposition 1.5. *$GKdim(A) \in \{0\} \cup \{1\} \cup [2, \infty]$ and for each of these numbers there is an algebra having this Gelfand Kirillov dimension.*

Proof. $GKdim(A) = 0$ if and only if A is finite dimensional as a vectorspace over k. If A is not finite dimensional, then $d_V(n) < d_V(n+1)$ for all n, hence $d_V(n) \succcurlyeq n$ and $GKdim(A) \geq 1$. Examples for Gelfand Kirillov dimension 0 and 1 are k and $k[x]$.
The fact that $GKdim(A) > 1$ implies $GKdim(A) \geq 2$ is known as Bergman's Gap Theorem ([2, Theorem 2.5] or [2, Section 12.2] for a proof using Ufnarovskii graphs).
As $GKdim(A[x]) = GKdim(A) + 1$ it suffices to give for any $\lambda \in]2, 3[$ an algebra A_λ with $GKdim(A_\lambda) = \lambda$. This follows from [2, Theorem 1.8] by choosing $f(x) = (x + 1)^\lambda$ (up to some scalar factor). $\square$

Quite recently the following result was proven in [3]:

Proposition 1.6. *There are no graded domains with GK-dimension strictly between 2 and 3.*

2 Ore-extensions

Proposition 2.1. *Let k be a field, A a k-algebra, $\sigma : A \to A$ a k-algebra morphism and $\delta : A \to A$ a $k-$linear $\sigma-$derivation. Then $GKdim(A[x, \sigma, \delta]) \geq GKdim(A) + 1$.*

Proof. Let V be a finite dimensional generating subspace for A, containing 1. Then $V + kx$ obviously is finite dimensional as well and it is a generating subspace for $A[x, \sigma, \delta]$.
Moreover for each $n \in \mathbb{N}, i \leq n$: $V^n x^i \subset V^{2n-i} x^i$, hence we have

$$V^n + V^n x + \ldots + V^n x^n \subset (V + kx)^{2n}$$

From this inclusion follows $(n+1)d_V(n) \leq d_{V+kx}(2n)$ and the inequality is immediate:

$$\begin{aligned} GKdim(A[x, \sigma, \delta]) &= \overline{\lim_{n\to\infty}} \log_n(d_{V+kx}(n)) \\ &= \overline{\lim_{n\to\infty}} \log_n(d_{V+kx}(2n)) \\ &\geq \overline{\lim_{n\to\infty}} \log_n((n+1)d_V(n)) \\ &= \overline{\lim_{n\to\infty}} \big(\log_n(n+1) + \log_n(d_V(n)) \big) \\ &= 1 + GKdim(A) \end{aligned}$$

□

Proposition 2.2. *Equality holds in the above Proposition if A admits a finite dimensional generating subspace V such that $\sigma(V) \subset V$.*

Proof. Let V be a finite dimensional generating subspace for A with $\sigma(V) \subset V$ and $1 \in V$. Since $A = \bigcup_{n=0}^{\infty} V^n$ there exists an $m \geq 1$ such that $\delta(V) \subset V^m$ and

$$\delta(V^{n+1}) \subset \sigma(V)\delta(V^n) + \delta(V)V^n$$

It follows by induction on n that $\delta(V^n) \subset V^{m+n}$. By induction we also show that

$$(V + kx)^n \subset V^{mn} + V^{mn}x + \ldots + V^{mn}x^n$$

This is obvious for $n = 0, 1$ and for the induction step remark that

$$\begin{aligned} V(V + kx)^n &\subset V^{mn+1} + V^{mn+1}x + \ldots + V^{mn+1}x^n \\ &\subset V^{m(n+1)} + V^{m(n+1)}x + \ldots + V^{m(n+1)}x^n \end{aligned}$$

and

$$\begin{aligned} x(V + kx)^n &\subset xV^{mn} + \ldots + xV^{mn}x^n \\ &\subset \sigma(V^{mn})x + \delta(V^{mn}) + \ldots + \sigma(V^{mn})x^{n+1} + \delta(V^{mn})x^n \\ &\subset V^{mn}x + V^{mn+m} + \ldots + V^{mn}x^{n+1} + V^{mn+m}x^n \\ &\subset V^{m(n+1)} + V^{m(n+1)}x + \ldots + V^{m(n+1)}x^n + V^{mn}x^{n+1} \end{aligned}$$

Hence $d_{V+kx}(n) \leq (n+1)d_V(mn)$ and

$$\begin{aligned} GKdim(A[x,\sigma,\delta]) &= \overline{\lim_{n\to\infty}} \log_n(d_{V+kx}(n)) \\ &\leq \overline{\lim_{n\to\infty}} \log_n((n+1)d_V(mn)) \\ &= \overline{\lim_{n\to\infty}} \big(\log_n(n+1) + \log_n(d_V(mn))\big) \\ &= 1 + \overline{\lim_{n\to\infty}} \log_n(d_V(n)) \\ &= 1 + GKdim(A) \end{aligned}$$ $\square$

Corollary 2.3. *Let* $A = \bigoplus_{n=0}^{\infty} A_n$ *be a graded* $k-algebra$ *where* $V = A_0 + A_1$ *is a finite dimensional generating subspace,* σ *a graded A-morphism of degree zero and* δ *a* σ*-derivation. Then* $GKdim(A[x,\sigma,\delta]) = GKdim(A) + 1.$

Examples.

- $GKdim(A[x_1,\ldots,x_m]) = GKdim(A) + m$, showing that the polynomial ring over k has Gelfand-Kirillov dimension equal to its number of variables.
- The n^{th} Weyl algebra A_n has Gelfand Kirillov dimension equal to $2n$. (recall $A_{n+1} \cong A_n[x_{n+1}][y_{n+1}, Id, \frac{\partial}{\partial x_{n+1}}]$)

References

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