ANAGRAMS: Dimension functions
Homological dimensions

Erik Rijcken

February 14, 2014

Introduction

These notes give a brief description on certain homological properties of modules over a ring, or more general, in an Abelian category. An object being projective, injective or flat is basically defined by the behavior of some widely-used homological functors: the Hom-functor and the tensor product. Similarly, how well one can approximate a given object by projective, injective or flat objects, will determine the behavior of the derived functors Ext and Tor.

The associated dimensions will try to summarize this information for the complete category of modules over a ring. Relations between the different dimensions will translate into relations between the corresponding functors. These notes briefly summarize some main results of [1], chapters 3, sections 6.2, 7.1-7.2, and chapter 8.

1 Projective and injective objects

1.1 Exactness of functors

Let $\mathcal{A}$ be an Abelian category (you can think of ${}_R\mathbf{Mod}$, the category of left R-modules[1]). Let F be a (covariant) functor from $\mathcal{A}$ to $\mathbf{Ab}$.

Definition 1.1. *We call F* left exact *when F preserves kernels, or equivalently, if for any exact sequence* $0 \longrightarrow A \longrightarrow B \longrightarrow C$ *, the sequence* $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$ *is exact.*

We call F right exact *when F preserves cokernels, or equivalently, if for any exact sequence* $A \longrightarrow B \longrightarrow C \longrightarrow 0$ *, the sequence* $F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$ *is exact.*

We call F exact *when F is both left and right exact, or equivalently, if F preserves short exact sequences.*

One can similarly define left exact, right exact, and exact for contravariant functors between Abelian categories. One can verify the following statement:

Proposition 1.2. *For any object A in $\mathcal{A}$, the covariant functor* $\mathsf{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to \mathbf{Ab}$ *and the contravariant functor* $\mathsf{Hom}_{\mathcal{A}}(-, A) : \mathcal{A} \to \mathbf{Ab}$ *are left exact.*

The question that follows naturally is: which conditions can be put on A such that (one of) these becomes an exact functor?

[1] By the Freyd-Mitchell embedding theorem, any (small) Abelian category can be embedded in ${}_R\mathbf{Mod}$ as a (full) subcategory, for some ring R [1, Theorem 5.99].

1.2 Projective objects

As one would say in Dutch, let's give the child a name, first focusing on the covariant Hom-functor:

Definition 1.3. *We call an object P* projective *if $\mathsf{Hom}_{\mathcal{A}}(P, -)$ is exact.*

There are of course several (fairly trivial) characterizations of this:

Proposition 1.4. *Let P be an object in $\mathcal{A}$. Then the following are equivalent:*

- *P is projective;*
- *$\mathsf{Hom}_{\mathcal{A}}(P, -)$ is right exact;*
- *when one has f and g as in the diagram below, there is always an h making the diagram commute:*

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\ M \arrow[r, two heads, "f"] & N \arrow[r] & 0 \end{tikzcd}$$

- *any short exact sequence ending in P splits.*

Example 1.5.

(a) When working in ${}_R\mathbf{Mod}$, we have $\mathsf{Hom}_R(R, M) \cong M$ by the identification $\varphi \mapsto \varphi(1)$, so $\mathsf{Hom}_R(R, -) \cong 1_{{}_R\mathbf{Mod}}$ is an exact functor. We conclude that R is projective. More general: any free module is projective.

(b) One can prove that P_1 and P_2 are projective if and only if $P_1 \oplus P_2$ is projective. Together with the first example, we find that summands of free modules are projective.[2]

(c) If $R = k$ is a field, all modules are projective, since they are all free.

When working in ${}_R\mathbf{Mod}$, every module is the quotient of a free module, and since free modules are projective, this means every module is the quotient of a projective module[3]. In general Abelian categories, this does not have to be the case.

Definition 1.6. *An Abelian category $\mathcal{A}$ has* enough projectives *if for every object A, there is a projective object P with an epimorphism $P \twoheadrightarrow A$.*

1.3 Injective objects

As we had projective objects, to indicate exactness of $\mathsf{Hom}_{\mathcal{A}}(P, -)$, we shall now have injective objects to indicate exactness of $\mathsf{Hom}_{\mathcal{A}}(-, E)$. This concept is dual to projective objects, so we can repeat our statement mutatis mutandis:

Definition 1.7. *We call an object E* injective *if $\mathsf{Hom}_{\mathcal{A}}(-, E)$ is exact.*

Proposition 1.8. *Let E be an object in $\mathcal{A}$. Then the following are equivalent:*

- *E is injective;*
- *$\mathsf{Hom}_{\mathcal{A}}(-, E)$ is right exact;*
- *when one has f and g as in the diagram below, there is always an h making the diagram commute:*

[2] Using for example the final equivalent statement for projective modules, one can also show that every projective module is a summand of a free module.

[3] For a (left) Artinian ring R and a finitely generated module M, there is a *projective cover*, in some sense a minimal projective module that projects onto M [2, Theorem 4.2].

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, hook, "f"] \arrow[d, "g"'] & N \arrow[dl, dashed, "h"] \\
& E &
\end{tikzcd}$$

- *any short exact sequence starting in E splits.*

One has a dual property to having enough projectives:

Definition 1.9. *An Abelian category $\mathcal{A}$ has* enough injectives *if for every object A, there is an injective object E with a monomorphism $A \hookrightarrow E$.*

It was fairly easy to show that ${}_R\mathbf{Mod}$ has enough projectives. Though the same statement holds for injectives, it is somewhat more involved to get.

Theorem 1.10. *For any ring R and any (left) R-module M, there is an injective R-module E in which one can embed M as a submodule, i.e. ${}_R\mathbf{Mod}$ has enough injectives.*

Proof. [1, Theorem 3.38] ☐

2 Projective and injective dimension

2.1 Resolutions

Definition 2.1. *Let A be an object in an Abelian category $\mathcal{A}$. A* projective resolution *of A is an exact sequence*

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

where $\{P_i\}_i \in \mathbb{N}$ are projective. An injective resolution *of A is an exact sequence*

$$\cdots \longrightarrow 0 \longrightarrow A \longrightarrow E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots$$

where $\{E_i\}_i \in \mathbb{N}$ are injective.
One constructs the deleted *projective (injective) resolution by removing A from the projective (injective) resolution.*

The existence of these resolutions is not always ensured; though in most 'nice' cases, one can find them.

Proposition 2.2. *If $\mathcal{A}$ has enough projectives (injectives), every object A has a projective (injective) resolution.*

Proof. Since $\mathcal{A}$ has enough projectives, we find a projective P_0 which projects onto A. We take the kernel of this projection, and find P_1 that projects onto it. By composing the projection maps and the kernel maps, we find a projective resolution:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[rr, "d_2"] \arrow[dr, two heads, "p_3"'] & & P_2 \arrow[rr, "d_1"] \arrow[dr, two heads, "p_2"'] & & P_1 \arrow[rr, "d_0"] \arrow[dr, two heads, "p_1"'] & & P_0 \arrow[r, "p_0"] & A \arrow[r] & 0 \arrow[r] & \cdots \\
& \ker p_2 \arrow[ur, hook] & & \ker p_1 \arrow[ur, hook] & & \ker p_0 \arrow[ur, hook] & & & &
\end{tikzcd}$$

One can make the same argument for an injective resolution, taking cokernels. ☐

Now we have defined these resolutions, you might ask, what are they good for? Well, the most basic thing one does with resolutions, is defining left and right derived functors [1, §6.2]. As an example, this is the procedure to define Ext. Assume we are working in an Abelian category $\mathcal{A}$ with enough injectives. We then define (for $n \geqslant 0$)

$$\mathsf{Ext}^n_{\mathcal{A}}(A, B) := H_{-n}(\mathsf{Hom}_{\mathcal{A}}(A, E_B)) \ ,$$

where E_B is a deleted injective resolution of B, and H_{-n} is the $(-n)$-th homology of the complex. To elaborate, we start by taking a deleted injective resolution[4]

$$E_B = \ \cdots \longrightarrow 0 \longrightarrow E_0 \xrightarrow{d^0} E_1 \xrightarrow{d^1} E_2 \xrightarrow{d^2} \cdots \quad ,$$

applying $\mathsf{Hom}_{\mathcal{A}}(A, -)$ to find

$$\mathsf{Hom}_{\mathcal{A}}(A, E_B) = \ \cdots \longrightarrow 0 \longrightarrow \mathsf{Hom}_{\mathcal{A}}(A, E_0) \xrightarrow{d^0_*} \mathsf{Hom}_{\mathcal{A}}(A, E_1) \xrightarrow{d^1_*} \mathsf{Hom}_{\mathcal{A}}(A, E_2) \xrightarrow{d^2_*} \cdots \quad ,$$

and lastly, we take the homology in the position of $\mathsf{Hom}_{\mathcal{A}}(A, E_n)$, to find

$$\mathsf{Ext}^n_{\mathcal{A}}(A, B) = \frac{\ker d^n_*}{\operatorname{im} d^{n-1}_*} \ .$$

Note 2.3. If $\mathcal{A}$ has both enough injectives and projectives, one can define $\mathsf{Ext}^n_{\mathcal{A}}(A, B)$ by using a projective resolution of A as well, and both methods give the same result [1, Theorem 6.67].

2.2 Dimensions

From the definition of the derived functors, one can see that it can be important to see when a projective or injective resolution becomes 0. In the example, as soon as $E_i = 0$, we have $\mathsf{Ext}^i_{\mathcal{A}}(A, B) = 0$. This illustrates the importance of the length of the resolutions, and justifies the next two definitions[5]:

Definition 2.4. *The* projective dimension *of a (left) R-module A is the smallest length of a projective resolution for A, i.e.*

$${}_R\mathrm{pd}(A) = \inf \left\{ n \in \mathbb{N} \;\middle|\; \begin{array}{l} \exists \textit{ a projective resolution } \ \cdots \to P_1 \to P_0 \to A \longrightarrow 0 \to \cdots \\ \textit{of } A, \textit{ with } \forall i > n : P_i = 0 \end{array} \right\} .$$

The injective dimension *of a (left) R-module A is the smallest length of an injective resolution for A, i.e.*

$${}_R\mathrm{id}(A) = \inf \left\{ n \in \mathbb{N} \;\middle|\; \begin{array}{l} \exists \textit{ an injective resolution } \ \cdots \to 0 \longrightarrow A \to E_0 \to E_1 \to \cdots \\ \textit{of } A, \textit{ with } \forall i > n : E_i = 0 \end{array} \right\} .$$

A simple observation regarding these matters: A is projective if and only if ${}_R\mathrm{pd}(A) = 0$, and A is injective if and only if ${}_R\mathrm{id}(A) = 0$. Looking into these matters in slightly more detail, one can prove the following:

Proposition 2.5. *Let A be a (left) R-module, then the following equivalences hold:*

$$\begin{aligned} {}_R\mathrm{pd}(A) \leqslant n \quad &\iff \quad \mathsf{Ext}^{n+1}_R(A, -) = 0 \\ &\iff \quad \forall k \geqslant n+1 : \mathsf{Ext}^k_R(A, -) = 0 \end{aligned}$$

$$\begin{aligned} {}_R\mathrm{id}(A) \leqslant n \quad &\iff \quad \mathsf{Ext}^{n+1}_R(-, A) = 0 \\ &\iff \quad \forall k \geqslant n+1 : \mathsf{Ext}^k_R(-, A) = 0 \end{aligned}$$

[4] One can show that the choice of resolution does not change the result of this procedure [1, Prop. 6.20, 6.40].

[5] From this point, we will work with R-modules. All these notions extend to Abelian categories with enough projectives (or/and injectives).

2.3 Global dimensions

So far, we have looked at projective and injective dimensions one module at a time. We can of course ask the question globally: how long can these resolutions become? The notion we introduce here, is the global dimension:

Definition 2.6. *The* left projective global dimension *of a ring R is*

$$\ell\mathrm{pD}(R) = \sup\left\{{}_R\mathrm{pd}(A) \;\middle|\; A \textit{ a left } R\textit{-module}\right\} .$$

The left injective global dimension *of a ring R is*

$$\ell\mathrm{iD}(R) = \sup\left\{{}_R\mathrm{id}(A) \;\middle|\; A \textit{ a left } R\textit{-module}\right\} .$$

When we look at the characterizations of ${}_R\mathrm{pd}$ and ${}_R\mathrm{id}$, we get the following rather surprising result:

Theorem 2.7. *For a ring R, the following equivalences hold:*

$$\begin{aligned} &\ell\mathrm{pD}(R) \leqslant n \\ \iff \quad & \forall k \geqslant n+1\ \forall A, B \in \mathrm{obj}\ {}_R\mathbf{Mod} : \mathsf{Ext}^k_R(A,B) = 0 \\ \iff \quad & \ell\mathrm{iD}(R) \leqslant n \end{aligned}$$

and thus $\ell\mathrm{pD}(R) = \ell\mathrm{iD}(R)$.

So our double definition actually give the same thing, which we will now call the left global dimension of R.

Definition 2.8. *The* left global dimension *of a ring R is $\ell\mathrm{D}(R) = \ell\mathrm{pD}(R) = \ell\mathrm{iD}(R)$.*

We now did everything for left R-modules. One could do the same for right R-modules, defining pd_R and id_R, and find the right global dimension $r\mathrm{D}(R)$. In general, these two global dimensions can be different.

Theorem 2.9. *If R is left and right Noetherian, $\ell\mathrm{D}(R) = r\mathrm{D}(R)$.*

Example 2.10. If $R = k$ is a field, all modules are projective, so $\ell\mathrm{D}(R) = 0$.

3 Flat dimension

3.1 Tensor product and Tor

Another basic functor in the category of R-modules, is the tensor product. As a reminder, this is the definition:

Definition 3.1. *Let A be a right R-module and B a left R-module. The* tensor product *is an Abelian group $A \otimes_R B$ and a R-biadditive*[6] *map $h : A \times B \to A \otimes_R B$, which is universal with this property.*

The universal property here means that for any other group G with a R-biadditive map f coming from $A \times B$, there is a unique map $\tilde{f}$ such that $f = h \circ \tilde{f}$. It implies that the tensor product is unique. If we denote $h(a,b) := a \otimes b$, one can prove that the tensor product is generated by these so-called simple tensors. Furthermore, if one takes the group represented by 'simple tensors', with the relations as given by R-biadditivity, one can prove that this is in fact the tensor product of A and B.

If we now fix B and let A vary, we get a map $- \otimes_R B : \mathbf{Mod}_R \to \mathbf{Ab}$, which is in fact a functor. If $f : A \to A'$ is a right R-homomorphism, we can define $f \otimes_R B$ as $f \otimes 1_B : a \otimes b \mapsto f(a) \otimes b$. One can ask about exactness of this functor and, similar to the Hom-functor, the $\otimes$-functor is always exact on one side. Proofs of the stated properties can be found in [1, §2.2].

[6] R-biadditive means: additive in both arguments, and $h(ar, b) = h(a, rb)$.

Proposition 3.2. *For any left R-module B, the (covariant) functor $- \otimes_R B$ is right exact.*

To deal with the question of left exactness of the tensor product, one looks at the left derived functor of $- \otimes_R B$, which is called Tor. This left derived functor is defined (for $n \geqslant 0$) as

$$\mathsf{Tor}_n^R(A, B) := H_n(P_A \otimes_R B) ,$$

where P_A is a deleted projective resolution of A. Similar to Ext, this procedure doesn't depend on the choice of projective resolution [1, Proposition 6.20]. One could have looked at the similar functor $A \otimes_R -$, and taken the left derived functor. Both procedures give the same result for $\mathsf{Tor}_n^R(A, B)$ [1, Theorem 6.32].

3.2 Flat modules

Again, one names the special modules, for which the tensor becomes exact.

Definition 3.3. *We call a (left) R-module F* flat *if $- \otimes_R F$ is exact.*

There are some equivalent characterizations of flatness:

Proposition 3.4. *Let F be a (left) R-module. Then the following are equivalent:*

- *F is flat;*
- *$- \otimes_R F$ is left exact;*
- *whenever $i : A \to A'$ is injective, so is $i \otimes 1_F : A \otimes_R F \to A' \otimes_R F$.*

Example 3.5.

(a) Since $A \otimes_R R \cong A$, we have $- \otimes_R R \cong 1_{\mathbf{Mod}_R}$, so R is flat.
(b) A direct sum of left R-modules is flat if and only if every summand is flat, essentially because the tensor product commutes with direct sums.
(c) The two previous examples prove that every summand of a free module is flat. Since these are exactly the projective modules, we get that all projective modules are flat (though not all flat modules are necessarily projective).

3.3 Flat resolutions

As in the case of projective and injective objects, we can now try to 'approximate' any module using flat modules.

Definition 3.6. *Let A be a left R-module. A* free resolution *of A is an exact sequence*

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

where $\{F_i\}_i \in \mathbb{N}$ are free.

One constructs the deleted free resolution *by removing A from the sequence.*

Luckily, since projective resolutions exist in ${}_R\mathbf{Mod}$, and projective modules are flat, the existence of free resolutions is guaranteed!

One could already suspect that these free resolutions will be relevant for the calculation of Tor. Although we defined Tor as a left derived functor, using projective resolutions, one could in fact use flat resolutions instead:

Theorem 3.7. *Let F_B be a deleted flat resolution of a left R-module B, and F_A a deleted flat resolution of a right R-module A. Then for all $n \geqslant 0$*

$$\mathsf{Tor}_n^R(A, B) = H_n(F_A \otimes_R B) = H_n(A \otimes_R F_B) .$$

Proof. [1, Theorem 7.5] □

3.4 Flat dimension

The previous theorem tells us that flat resolutions relate to Tor as projective and injective resolutions relate to Ext. In particular, the length of flat resolutions gives us information on when Tor will be guaranteed to become zero. This leads to the introduction of a new dimension for modules:

Definition 3.8. *The* flat dimension *of a (left) R-module B is the smallest length of a flat resolution for B, i.e.*

$$ {}_R\mathrm{fd}(B) = \inf \left\{ n \in \mathbb{N} \;\middle|\; \begin{array}{l} \exists \textit{ a flat resolution } \cdots \to F_1 \to F_0 \to B \longrightarrow 0 \to \cdots \\ \textit{of } A, \textit{ with } \forall i > n : F_i = 0 \end{array} \right\} . $$

Similar to the projective and injective dimension, one gets: B is flat if and only if ${}_R\mathrm{fd}(B) = 0$. A more rigorous analysis of the situation gives the following:

Proposition 3.9. *Let B be a left R-module, then the following equivalences hold:*

$$ \begin{aligned} {}_R\mathrm{fd}(B) \leqslant n \quad &\Longleftrightarrow \quad \mathsf{Tor}^R_{n+1}(-, B) = 0 \\ &\Longleftrightarrow \quad \forall k \geqslant n+1 : \mathsf{Tor}^R_k(-, B) = 0 \end{aligned} $$

Lastly, since projective modules are flat, any projective resolution is a flat resolution, so

Proposition 3.10. *For any left R-module B, ${}_R\mathrm{fd}(B) \leqslant {}_R\mathrm{pd}(B)$.*

3.5 Weak dimension

Similarly to the global dimension, we can combine all flat dimensions into one dimension, depending on the ring:

Definition 3.11. *The* left weak dimension *of a ring R is*

$$ \ell\mathrm{wD}(R) = \sup \left\{ {}_R\mathrm{fd}(B) \;\middle|\; B \textit{ a left } R\textit{-module} \right\} . $$

At this point, we have looked at left R-modules B, and the corresponding functors $- \otimes_R B$ and $\mathsf{Tor}^R_n(-, B)$ for $n \geqslant 0$. Looking similarly to right R-modules A, $A \otimes_R -$ and $\mathsf{Tor}^R_n(A, -)$ for $n \geqslant 0$, we can define $\mathrm{fd}_R(A)$ and the following:

Definition 3.12. *The* right weak dimension *of a ring R is*

$$ r\mathrm{wD}(R) = \sup \left\{ \mathrm{fd}_R(A) \mid A \textit{ a right } R\textit{-module} \right\} . $$

We would get a similar statement to Proposition 3.9, and putting those together, we get

Theorem 3.13. *For a ring R, the following equivalences hold:*

$$ \begin{aligned} &\ell\mathrm{wD}(R) \leqslant n \\ &\quad \Longleftrightarrow \quad \forall k \geqslant n+1 \; \forall A \in \mathrm{obj}\, \mathbf{Mod}_R \; \forall B \in \mathrm{obj}\, {}_R\mathbf{Mod} : \mathsf{Tor}^R_k(A, B) = 0 \\ &\quad \Longleftrightarrow \quad r\mathrm{wD}(R) \leqslant n \end{aligned} $$

and thus $\ell\mathrm{wD}(R) = r\mathrm{wD}(R)$.

Similar to the global dimension, we can reduce our two dimensions to one:

Definition 3.14. *The* weak dimension *of a ring R is $\mathrm{wD}(R) = \ell\mathrm{wD}(R) = r\mathrm{wD}(R)$.*

4 Connections and properties

4.1 Finding dimensions

In a few cases, one can find the dimensions faster, if one has some more information on the ring. The following statement says that it's sufficient to only look at quotients of the ring itself:

Theorem 4.1. *If R is a ring, we have*

$$\ell\mathrm{D}(R) = \sup\left\{ {}_R\mathrm{pd}(R/I) \mid I \text{ a left ideal} \right\} ;$$
$$\mathrm{wD}(R) = \sup\left\{ {}_R\mathrm{fd}(R/I) \mid I \text{ a left ideal} \right\} .$$

Secondly, if one adds an indeterminate to a ring, the global dimension increases by one:

Theorem 4.2 (Hilbert's Syzygy Theorem)**.** *If R is a ring, we have*

$$\ell\mathrm{D}(R[x_1, \ldots, x_n]) = \ell\mathrm{D}(R) + n .$$

In particular, if k is a field, $\ell\mathrm{D}(k[x_1, \ldots, x_n]) = n$.

Proofs of these properties can be found in [1, §8.1-8.2].

4.2 Weak versus global dimension

Due to projective modules being flat, we already observed that ${}_R\mathrm{fd}(A) \leqslant {}_R\mathrm{pd}(A)$, in general. Ranging over all modules A, this gives the following property:

Proposition 4.3. *For any ring R, we have* $\mathrm{wD}(R) \leqslant \min(\ell\mathrm{D}(R), r\mathrm{D}(R))$.

When we combine the alternative characterizations of weak and global dimension, in terms of Ext and Tor, this gives the following:

Corollary 4.4. *If $\mathsf{Ext}^k_R(A, B) = 0$ for all left (or all right) R-modules A, B, then $\mathsf{Tor}^R_k(C, D) = 0$ for all right R-modules C and all left R-modules D.*

In a few specific cases, we have equality in the above proposition:

Proposition 4.5. *If R is left Noetherian,* $\mathrm{wD}(R) = \ell\mathrm{D}(R)$.

It will not surprise you that the dual statement also holds, if R is right Noetherian, $\mathrm{wD}(R) = r\mathrm{D}(R)$. Together, these give

Corollary 4.6. *If R is left and right Noetherian,* $r\mathrm{D}(R) = \mathrm{wD}(R) = \ell\mathrm{D}(R)$.

Proofs of these properties can be found in [1, §8.1].

4.3 Commutative Noetherian local rings

In the specific case where R is commutative and Noetherian, we already have a somewhat more special property (not forgetting that in this case $\ell\mathrm{D}(R) = r\mathrm{D}(R)$).

Theorem 4.7. *If R is a commutative Noetherian ring, we have*

$$\ell\mathrm{D}(R) = \sup\{\ell\mathrm{D}(R_{\mathfrak{m}}) \mid \mathfrak{m} \text{ a maximal ideal}\} .$$

When we add the requirement that R is local, this supremum is running over a single maximal ideal, so $\ell\mathrm{D}(R) = \ell\mathrm{D}(R_{\mathfrak{m}})$, and furthermore, one can look at the residue field:

Theorem 4.8. *If R is a commutative Noetherian local ring with residue field k, we have*

$$\ell\mathrm{D}(R) = {}_R\mathrm{pd}(k) \ .$$

We can also compare the global dimension to the Krull dimension[7] $\dim(R)$. When the ring is 'nice', these two are equal:

Theorem 4.9. *Let R be a commutative Noetherian local ring. Then R is regular[8] if and only if $\ell\mathrm{D}(R)$ is finite, and in that case $\ell\mathrm{D}(R) = \dim(R)$.*

Proofs of these properties can be found in [1, §8.4].

References

[1] Joseph J. Rotman. *An introduction to homological algebra.* Universitext. Springer, New York, second edition, 2009.

[2] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.

[7] The Krull dimension of a ring is the maximal length of a strictly increasing chain of prime ideals.

[8] A local ring $(R, \mathfrak{m}, k)$ is regular if it's Krull dimension equals $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$