KRULL DIMENSION

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1 Dimension

Throughout this talk, R is a commutative ring.

Definition. Let P be a prime ideal of R. We say that P has height t and write $ht(P) = t$ if there exists a chain of prime ideals

$$\emptyset = P_0 \subsetneq \cdots \subsetneq P_t = P$$

but no longer chain.

We define

$$\dim(R) = \sup\{ht(P),\ P \in Spec(R)\},$$

called the *Krull dimension* of R.

Properties:

- $ht(S^{-1}P) = ht(P)$,
- $ht(P) + \dim(R/P) \leq \dim R$,
- $\dim(R) + 1 \leq \dim(R[X]) \leq 2\dim(R) + 1$.

Example:

- $\dim(\mathbb{Z}) = 1,\ \dim(\mathbb{Z}[X]) = 2$,
- Let K be an arbitrary field. We have
  - $\dim(K[X_1, \ldots, X_n]) = n$,
  - $R = K \times \mathbb{Z}[X] \implies \begin{cases} \dim(R) = 2 \text{ (Ex: } 0 \subsetneq K \times (2) \subsetneq K \times (2, X)), \\ \dim(R/P) = 0,\ ht(P) = 0 \text{ where } P = 0 \times \mathbb{Z}[X]. \end{cases}$

Known results

1. Let $R \subset S$ be commutative rings. If S is integral over R then $\dim(S) = \dim(R)$.

2. If R is a Noetherian ring then $\dim(R[X]) = \dim(R) + 1$ and so

$$\dim(R[X_1, \ldots, X_n]) = \dim(R) + n.$$

2 Transcendence degree

Definition 2.1. Let $\alpha_1, \ldots, \alpha_n \in R$. We say that $\alpha_1, \ldots, \alpha_n \in$ are *algebraically independent over* R if there is no non-trivial polynomial in n variables of coefficients in R vanishing on $(\alpha_1, \ldots, \alpha_n)$.

Lemma 2.2. *Let $F \subset K$ be field extension and let $\alpha_1, \ldots, \alpha_n \in K$. Then $\alpha_1, \ldots, \alpha_n$ are algebraically independent over F iff α_i is transcendental over $F(\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \alpha_n)$ for any $i = 1, \ldots, n$.*

Lemma 2.3. *Let $F \subset K$ be field extension. Let $\alpha_1, \ldots, \alpha_n \in K$ be algebraically independent over F and let $\beta_1, \ldots, \beta_m \in K$ be algebraically independent over F. Assume that α_i is algebraic over $F(\beta_1, \ldots, \beta_m)$ for each $i = 1, \ldots, n$ and that β_j is algebraic over $F(\alpha_1, \ldots, \alpha_n)$ for each $j = 1, \ldots, m$. Then we have $n = m$.*

Definition 2.4. Let $F \subset K$ be field extension. If $\alpha_1, \ldots, \alpha_n$ are algebraically independent and K is algebraic on $F(\alpha_1, \ldots, \alpha_n)$ then we say that $(\alpha_1, \ldots, \alpha_n)$ is a *transcendental basis of K over F* and that K has transcendence degree n, denote $\text{tr.deg}_F K := n$.

Properties: Let $F \subset K \subset L$ be field extension. Then $\text{tr.deg}_F L = \text{tr.deg}_K L + \text{tr.deg}_F K$.

Noether's Normalization: Let A be an affine algebra over a field K. Then there exist a tuple $(y_1, \ldots, y_n)$ of elements in A such that $y_1, \ldots, y_n$ are algebraically independent over K and A is integral over $K[y_1, \ldots, y_n]$. If $(x_1, \ldots, x_m)$ is also such a tuple, then we have $n = m$ and $\dim(A) = \text{tr.deg}_K A = n$.

Theorem 2.5. *Let A be an affine algebra over a field K. Assume that A is an integral domain. If M is a maximal ideal of A then we have $\dim(M) = \dim(A)$. Consequently, we have*

$$ht(P) + \dim(A/P) = \dim(A) \quad \textit{for} \quad P \in Spec(A)$$

3 Dimension of Schemes

Definition 3.1. Let X be a topological space. We say that X has dimension n and write $\dim(X) = n$ if there exists a chain of irreducible closed subsets of X

$$\emptyset = Z_0 \subsetneq \cdots \subsetneq Z_n$$

but no longer chain.

Remark 3.2.

(i) $\dim(\mathbb{A}^n_K) = n$ since

$$\emptyset \subsetneq V(X_1, \ldots, X_n) \subsetneq V(X_1, \ldots, X_{n-1}) \subsetneq \ldots \subsetneq V(X_1) \subsetneq \mathbb{A}^n_k$$

(ii) For any ideal I of $K[X_1, \ldots, X_n]$, the vanishing set $V(I)$ is irreducible iff I is prime. Therefore, if Y is an algebraic set then Y, viewed as a topological space, has dimesion

$$\dim(Y) = \dim(K[X_1, \ldots, X_n]/\mathcal{I}(Y)).$$

Moreover, if Y is a variety, then

$$\dim(Y) = \text{tr.deg}_K(K[X_1, \ldots, X_n]/\mathcal{I}(Y)).$$

(iii) For any commutative ring A, if $X = \mathit{Spec}(A)$, then we have

$$\dim(X) = \dim(A).$$

Theorem 3.3 (Krull's Principal Ideal Theorm)**.** *Let R be a commutaive Noetherian ring and let $a \in R$ be a non-unit. Let P be a minimal prime ideal of the principal ideal (a) of R. Then $ht(P) \leq 1$. If a is not a non-zero divisor, then $ht(P) = 1$.*

Remark 3.4. Let f be a polynomial of $K[X_1, \ldots, X_n]$. Assume that $\sqrt{f} = P_1 \cap \ldots P_k$ is a decomposition of minimal prime ideals. Then $\dim(P_i) = 1$ and so $\dim(V(P_i)) = n - 1$ and $\dim(V(f)) = n - 1$.

Theorem 3.5 (Krull's Generalized Principal Ideal Theorm)**.** *Let R be a commutative Noetherian ring and let I be a proper ideal of R which can be generated by n elements. Then $ht(P) \leq n$ for each minimal prime ideal P of I.*

Definition 3.6. Let X be a topological space and $Y \subset X$. We say that Y has codimension n and write $\mathit{codim}_X(Y) = n$ if there exists a chain of irreducible closed subsets of X

$$Y = Y_0 \subsetneq \cdots \subsetneq Y_n \subsetneq X$$

but no longer chain.

It is obvious that

$$codim_X(Y) + \dim(Y) \leq \dim(X).$$

When does the equality hold ?

Lemma 3.7. *Let X be a topological space.*

(i) $\dim(X) = \sup_\alpha \dim(U_\alpha)$ *for any cover* $X = \cup_\alpha U_\alpha$.

(ii) If X is a scheme then $\dim(X) = \sup\{\dim(\mathcal{O}_{X,x}),\ x \in X\}$.

Theorem 3.8. *Let X be an integral scheme of finite type over field k with its function field K. Then*

(1) $\dim(X) = \text{tr.deg}_k(K) < \infty$.

(2) For any open subset $U \subset X$, we have $\dim(X) = \dim(U)$.

(3) For any closed point $p \in X$, we have $\dim(X) = \dim(\mathcal{O}_{X,p})$.

(4) Let Y be irreducible closed subset of X. Then

$$\dim(Y) + codim_X(Y) = \dim(X).$$

Example: Let $X = Spec(k \times \mathbb{Z}$. Take $Y = p = 0 \times \mathbb{Z}$. Then

$$2 = \dim(X) \neq \dim(\mathcal{O}_{X,p}) = 0$$

and

$$\dim(Y) + codim(Y) = 0 < \dim(X).$$