ANAGRAMS: Dimension of triangulated categories

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Abstract

These notes were prepared for the first session of "Dimension functions" of ANAGRAMS seminar. The goal is to introduce the dimension of a triangulated category and outline some of its properties. Examples of finite-dimensional triangulated categories with geometric relevance will be sketched. These examples will provide relations with other dimension functions that are treated in the seminar.

Contents

1 What is a triangulated category?

1.1 History

Triangulated categories were introduced by Verdier in his thesis which was presented in 1967. The setting was the introduction of derived categories as it has been sketched in the previous lecture on Grothendieck duality. Recall that we start with an abelian category and we consider the category of complexes. By identifying homotopic morphisms of complexes we get the homotopy category. This category is not abelian any more, but it still has some structure derived of its abelian origin. This structure is what Verdier studied and axiomatized, and

he defined a triangulated category as any category having this kind of structure. Afterwards, by inverting quasi-isomorphisms, the derived category of the abelian category is obtained, which is again a triangulated category.
Hence from Verdier's thesis we obtain the first two examples of triangulated categories. However, there are some other categories that carry a natural triangulated structure in other settings, which makes it worthwhile studying them from a general and abstract point of view.
Though the notion of triangulated category is due to Verdier, a similar set of axioms[1] was outlined by Dold and Puppe in the 60's in the setting of algebraic topology.

1.2 Definition and axioms

Recall that an additive category is a category with the properties:

- Homomorphism sets have abelian group structure such that composition is bilinear.
- Both the direct sum and the product of any finite set of objects exist and they coincide.

Definition 1. A triangulated category $\mathcal{T}$ is an additive category together with an auto-equivalence $[1] : \mathcal{T} \to \mathcal{T}$ which is called shift functor[2], and a fixed set of diagrams of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \tag{1}$$

which are called triangles, and where morphisms between two triangles are given by morphisms u, v, w such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[r, "h"] \arrow[d, "w"] & X[1] \arrow[d, "u[1]"] \\ X' \arrow[r, "f"] & Y' \arrow[r, "g"] & Z' \arrow[r, "h"] & X'[1] \end{tikzcd} \tag{2}$$

is commutative.
Moreover, the following axioms hold:

TR1
- Every diagram of the form of (1) isomorphic to a triangle is a triangle.
- The diagram $X \xrightarrow{Id_X} X \xrightarrow{0} 0 \xrightarrow{0} X[1]$ is a triangle.

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[1]The octahedral axiom, which is one of Verdier's axioms, does not appear in Dold and Puppe's set.

[2]We are giving the abstract definition of a triangulated category, but some of the terminology recalls us the setting where Verdier introduced them. That is the case of the "shift functor", as in derived categories the actual shift functor in complexes is the one playing this role.

- Any morphism $f : X \to Y$ of objects in $\mathcal{T}$ can be embbeded in a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ where Z is called the cone[3] of the morphism f and usually denoted $\mathsf{Cone}(f)$.

TR2 The diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a triangle if and only if $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is a triangle.

TR3 Given two triangles, $X \to Y \to Z \to X[1]$ and $X' \to Y' \to Z' \to X'[1]$, and two morphisms, $u : X \to X'$ and $v : Y \to Y'$, such that the first square in

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "u[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd} \tag{3}$$

is commutative, then there is a morphism $w : Z \to Z'$, not necessarily unique[4], completing the diagram such that u, v, w form a morphism of triangles.

TR4 The octahedral axiom. The idea of the axiom is that given three triangles with some of their objects in common, there exists a triangle which fits perfectly with the previous three providing a nice commutative diagram. As it is a rather technical axiom when stated, we will not go through it here. The exact formulation can be found in any reference dealing with triangulated categories. However it is worthwhile mentioning that there is a nice interpretation that allows us to see it as an analogue on triangulated categories for Noether's Second Isomorphism Theorem in abelian categories.
One may wonder which is the motivation for the name of the axiom. The reason is that one of the ways of representing the triangles involved takes the form of an octahedron. However, other graphical representations can be found in different references for example in the form of a braid or of a square.

Notation 2. The name *triangle* for these diagrams comes from one of the usual ways of representing them. A triangle $X \to Y \to Z \to X[1]$ in our notation is usually displayed

$$\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z \arrow[ul, "+1"] & \end{tikzcd} \tag{4}$$

where $+1$ indicates which is the morphism that maps to the shifted object.

[3]The name of cone is given after the cone complex, which plays the role of the cone in the setting of the homotopy category of an abelian category.

[4]The lack of uniqueness of this morphism tells us that the operation of taking cones is not functorial, which is one of the main issues when dealing with triangulated categories. This problem leads some authors to doubt whether the notion we use of triangulated category is the "correct" one.

Notation 3. The shift functor applied a number n of times will be denoted by $[n]$, i.e. $[n] = [1] \circ \ldots \circ [1]$.
The inverse of the shift functor is denoted $[-1]$ (recall that the shift functor is an equivalence) and composed with itself n times will be denoted $[-n]$.

Remark 4. Verdier's axioms have been analysed deeply and some variations of them can be found in certain references. For example in [6], the octahedral axiom is replaced by an equivalent statement. Moreover, in [5] it is proven that we do not need to require all of them, as **TR3** and the inverse implication of **TR2** can be deduced from the others. Despite the fact, they are usually written including these "redundant" parts. The main reason is to keep things simple; if we did not consider them as part of the axioms, they would need to be stated as results immediately after, as they are required for working with triangulated categories from the beginning.

1.3 Examples of triangulated categories

We already have dealt with two examples of triangulated categories: the homotopy category of an additive category and the derived category of an abelian category. These are the ones that actually were a model for the definition. Nevertheless, it turns out that this structure appears in other categories in a natural way such as:

- The category of perfect complexes over an additive category.
- Stable homotopy category in topology.
- Stable module category in representation theory.
- Homotopy category of a stable ∞-category.
- ...

2 Generators of triangulated categories. Dimension function

The dimension of a triangulated category comes up as a way of measuring how fast one can recover the category by doing certain operations with a single object, which is the generator of the category. However, some different definitions of generators can be given for a triangulated category. We proceed now to review all of them, including the one we are interested in, and we will give the motivations for the choice.

2.1 Generators of a triangulated category

First of all we will need some previous definitions and notations.

Definition 5. Let $\mathcal{T}$ be a triangulated category. A subcategory $\mathcal{A}$ is called *thick* or *épaisse* if it is closed under isomorphisms and direct summands.

Definition 6. Let $\mathcal{T}$ be a triangulated category. A subcategory $\mathcal{A}$ is called *dense* if every object of $\mathcal{T}$ is isomorphic to a direct summand of an object in $\mathcal{A}$.

Notation 7. Let $\mathcal{E}$ be a set of objects of a triangulated category $\mathcal{T}$.

- $\mathsf{add}\,(\mathcal{E})$ is the minimal full subcategory of $\mathcal{T}$ which contains $\mathcal{E}$ and is closed under finite direct sums and shifts.
- $\mathsf{smd}\,(\mathcal{E})$ is the minimal full subcategory of $\mathcal{T}$ which contains $\mathcal{E}$ and is closed under finite direct summands (whenever it is possible to take them).
- We denote by $\langle \mathcal{E} \rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{E}$ and closed under finite direct sums, direct summands and shifts, i.e. $\langle \mathcal{E} \rangle = \mathsf{smd}\,(\mathsf{add}\,(\mathcal{E}))$.

Let $\mathcal{A}, \mathcal{B}$ be two subcategories of the category $\mathcal{T}$.

- We denote by $\mathcal{A} \star \mathcal{B}$ the full subcategory of $\mathcal{T}$ consisting of objects X which occur in a triangle $A \to X \to B \to A[1]$ where A is an object in $\mathcal{A}$ and B is an object in $\mathcal{B}$. By doing an analogy with abelian categories, we could say is the full category consisting of extensions of objects in $\mathcal{B}$ by objects in $\mathcal{A}$.
- We set $\mathcal{A} \diamond \mathcal{B} = \langle \mathcal{A} \star \mathcal{B} \rangle$.
- We write $\langle \mathcal{A} \rangle_0 = 0$ and then we define by induction $\langle \mathcal{A} \rangle_n = \langle \mathcal{A} \rangle_{n-1} \diamond \langle \mathcal{A} \rangle$. Thus objects in $\langle \mathcal{A} \rangle_n$ are summands of objects obtained by computing n extensions of finite direct sums and shifts of objects in $\mathcal{A}$.
  We write $\langle \mathcal{A} \rangle_\infty = \bigcup_{n \geq 0} \langle \mathcal{A} \rangle_n$.

Once we have set the notations, we can define the different types of generation of a triangulated category one can find in the literature.

Definition 8. Let $\mathcal{E}$ be a set of objects in the triangulated category $\mathcal{T}$.

- We say $\mathcal{E}$ *generates* $\mathcal{T}$ if the following property holds: If we have an object X in $\mathcal{T}$ such that $\mathrm{Hom}(E[n], X) = 0$ for all E in $\mathcal{E}$ and all n (this is the definition for the object X to be in the orthogonal of $\mathcal{E}$), then $X = 0$. In other words, we can say that $\mathcal{E}$ generates $\mathcal{T}$ if $\mathcal{E}^\perp = 0$, where $\mathcal{E}^\perp$ is the orthogonal of $\mathcal{E}$.
- We say $\mathcal{E}$ *classically generates* $\mathcal{T}$ if the smallest thick subcategory of $\mathcal{T}$ which contains $\mathcal{E}$ is $\mathcal{T}$ itself. This is equivalent to saying that $\mathcal{T} = \langle \mathcal{E} \rangle_\infty$.
  We say $\mathcal{T}$ is *classically finitely generated* if it is classically generated by a single object.
- We say $\mathcal{E}$ *strongly generates* $\mathcal{T}$ if there exists an integer $d < \infty$ such that $\mathcal{T} = \langle \mathcal{A} \rangle_d$.
  We say $\mathcal{T}$ is *strongly finitely generated* if it is strongly generated by a single object.

Remark 9. Notice that each definition is strictly stronger than the previous, so strongly generated implies classically generated and this last one implies generated, however implications in the other way are not true.

Remark 10. Observe that if the triangulated category is classically generated (resp. strongly generated) by a finite collection of objects, it is classically finitely generated (resp. strongly finitely generated), as the direct sum of the finite collection of generators is again a generator.

Remark 11. One can find often the term *compactly generated* for an additive category, thus for a triangulated category too. This means that it is generated by a collection of compact objects. The notion is important, for example, for the analysis of categories which do not admit arbitrary direct sums (see [9]). But we will not discuss this here any further.

2.2 Definition of the dimension and motivations

Let $\mathcal{T}$ be a triangulated category.

Definition 12. The *dimension* of $\mathcal{T}$ is the minimal integer d such that there exists an object X in $\mathcal{T}$ with $\langle X \rangle_{d+1} = \mathcal{T}$.
If there is not such an object, we say the dimension is ∞.

This definition is due to Rouquier [9], who was inspired by the Bondal and Van den Bergh paper [3] where the definition of a strong generator of a triangulated category was given for the first time. Bondal and Van den Bergh found out that the existence of such a generator in an Ext-finite triangulated category provides a sufficient condition for every contravariant cohomological functor of finite type for being representable. As this result escapes from our goal for this seminar we will not explain it in exact terms. The main idea is that if a triangulated category of certain type is strongly finitely generated, then certain functors of great relevance on it have the always desirable property of being representable. Moreover, the authors provide the first examples of triangulated categories of finite dimension, where they could apply their representability result. Hence in their article we can find the first and one of the main results related to the finiteness of the dimension of a triangulated category, the motivation of its definition, and our first examples of geometric relevance.

After defining the dimension in [9], Rouquier took further the analysis of the dimension of triangulated categories of geometric importance, providing us with a huge bunch of results that relate this new dimension to other geometrical and homological dimensions already known, with applications in algebraic geometry and representation theory. Other authors since then have provided more examples and bounds for the dimension of certain triangulated categories and other useful applications, such as Aihara and Takahashi, Krausse, Opperman, Orlov... Some of the most interesting examples will be stated in the last section.

It is important to remark that the definition of dimension is really young, as it dates from 2008. Consequently, the study of its properties and applications is a field where many other contributions can be still done.

Remark 13. In [9], Rouquier gives a list of other possible "dimensions" of triangulated categories that could be of interest. Given that the definition already provided (and sometimes known as Rouquier dimension) has been truly fruitful in applications and results, we will not go through the list and we will focus only on the definition already stated.

2.3 First general properties

In [9], we can find the first properties of the dimension in an arbitrary triangulated category.

Proposition 14. *Let $\mathcal{T}$ be a triangulated category.*

P1 *If $\mathcal{A}$ is a dense full triangulated subcategory of $\mathcal{T}$, then* $\dim \mathcal{A} = \dim \mathcal{T}$.

P2 *Let $\mathcal{T}'$ be another triangulated category and let $F : \mathcal{T} \to \mathcal{T}'$ a triangulated functor with dense image. Then if $\mathcal{T} = \langle \mathcal{A} \rangle_d$, we have that $\mathcal{T}' = \langle F(\mathcal{A}) \rangle_d$. Consequently,* $\dim \mathcal{T}' \leq \dim \mathcal{T}$.
In particular, if $\mathcal{A}$ is a thick subcategory of $\mathcal{T}$, we have that $\dim \mathcal{T}/\mathcal{A} \leq \dim \mathcal{T}$.

P3 *Let $\mathcal{A}, \mathcal{B}$ be two triangulated subcategories of $\mathcal{T}$ such that $\mathcal{A} \diamond \mathcal{B} = \mathcal{T}$. Then* $\dim \mathcal{T} \leq 1 + \dim \mathcal{A} + \dim \mathcal{B}$.

P4 *Dimension behaves well with respect to taking the opposite category, i.e.* $\dim \mathcal{T} = \dim \mathcal{T}^{op}$.

Remark 15. In particular, **P2** tells us that dimension works with "surjections" in the nice way we are used to and from **P3** one can also deduce that the dimension behaves somehow additively in relation to the operation $\diamond$.

As the goal of this seminar is providing the properties and statements related to a dimension function and not giving the proofs, we will skip the proof of this proposition as well as all the proofs of the statements we will go through below.

3 Examples of finite dimensional triangulated categories

Along this section some interesting results that can be obtained when studying the dimension of the derived categories of sheaves and rings will be provided. In many of these results, the dimension will be bounded by another dimension function, related to the rings and the sheaves we start with. This way, many nice relations between some of the functions that we will deal with in this seminar and the dimension of these derived categories will be outlined, proving that dimension in this case carries some geometric information that remained hidden at first sight.
After stating these results, two different applications of the dimension, one in mirror symmetry theory and one in representation theory, will be sketched.

3.1 Dimension of derived categories of rings and sheaves

Derived categories and categories of perfect complexes are examples of triangulated categories built over an abelian category.

On one hand, depending on the starting data, one can prove that the dimension of the derived category is finite, and in some cases bounds can be given. These bounds will come in form of other dimension functions, which may increase the interest in the relations among the talks in this seminar.

On the other hand, the dimension of the category of perfect complexes over a ring or a scheme will also provide information about regularity of the ring or the scheme.

Along the statement of the results some terminology related to representation theory or sheaf theory will come up. The definitions can be found in any basic manual on the subjects.

Derived categories of modules over a ring

Given a ring A, we will consider the derived category of finitely generated A-modules, denoted by $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$. We have:

- Let A be a ring of finite global dimension
  - If A is noetherian
    $$\dim \mathbf{D}^{\mathrm{b}}(A\text{-mod}) \leq 1 + 2\mathrm{gl.dim}\, A$$
  - If A is a finite dimensional algebra over a field k, then
    $$\dim \mathbf{D}^{\mathrm{b}}(A\text{-mod}) \leq \mathrm{gl.dim}\, A$$
    This result can be also extended to dg algebras after defining global dimension in that setting, which can be done in a natural way.
  - We can get an improvement for the first statement when we also force A to be hereditary, in addition to be noetherian. Then
    $$\dim \mathbf{D}^{\mathrm{b}}(A\text{-mod}) = 1$$
    and a generator of the category is A itself considered as an $A-\mathrm{mod}$.
- Let A be an Artin ring. Then
  $$\dim \mathbf{D}^{\mathrm{b}}(A\text{-mod}) \leq \mathrm{ll}\,(A) - 1$$
  where $\mathrm{ll}\,(A)$ denotes the Loewy length of the ring A, i.e. the small integer i such that $\mathbf{r}^i A = 0$ where $\mathbf{r}$ is the radical of A.
- Let A be a commutative complete noetherian local ring, then
  $$\dim \mathbf{D}^{\mathrm{b}}(A\text{-mod}) < \infty$$

Derived categories of quasi-coherent and coherent sheaves over a scheme

Given a scheme X, we will consider the bounded derived categories of quasi-coherent and coherent $\mathcal{O}_X$-modules, denoted by $\mathbf{D}^{\mathrm{b}}(X\text{-Qcoh})$ and $\mathbf{D}^{\mathrm{b}}(X\text{-coh})$ respectively. We have:

- Let X be a smooth quasi-projective scheme over a field k. Then
  $$\dim \mathbf{D}^{\mathrm{b}}(X\text{-Qcoh}) \leq 2\dim X$$
  $$\dim \mathbf{D}^{\mathrm{b}}(X\text{-coh}) \leq 2\dim X$$
  Where $\dim X$ denotes the (Krull) dimension of the scheme. As an example, from Beilinson's result on [2], it can be deduced that
  $$\dim \mathbf{D}^{\mathrm{b}}(\mathbf{P}^n\text{-coh}) = n$$
- Let X be a reduced separable scheme of finite type over a field k, then
  $$\dim \mathbf{D}^{\mathrm{b}}(X\text{-coh}) \geq \dim X$$

- Given a smooth affine scheme X of finite type over k one has

$$\mathsf{dim}\,\mathbf{D}^{\mathrm{b}}(X\text{-coh}) = \mathsf{dim}\,X = \mathrm{Krull.dim}\mathcal{O}_X = \mathsf{gl.dim}\,\Gamma(X, \mathcal{O}_X)$$

- Let X be a regular quasi-projective scheme over k, then

$$\mathsf{dim}\,\mathbf{D}^{\mathrm{b}}(X\text{-coh}) \leq 2(1 + \mathsf{dim}\,X)^2 - 1$$

- This last result was first improved by Rouquier in [9] for the case when X is chosen also to be a curve. Then

$$\mathsf{dim}\,\mathbf{D}^{\mathrm{b}}(X\text{-coh}) \leq 3$$

  which is a bound considerably smaller than 7, the one we would get using the last result.
  However, Orlov in [7] could reduce this last bound for a curve and, moreover, he proved the equality

$$\mathsf{dim}\,\mathbf{D}^{\mathrm{b}}(X\text{-coh}) = 1$$

  In fact, his result is more general, as he proved the statement for the relative case (X a smooth quasi-projective curve).
  This result shows us that the dimension turns to have real geometric importance. Furthermore, Orlov also conjectured that this result holds in general for higher dimensions, i.e. if X is a smooth quasi-projective scheme of dimension n, then $\mathsf{dim}\,\mathbf{D}^{\mathrm{b}}(X\text{-coh}) = n$.

- Let X be a separeted scheme of finite type over a perfect field k, then

$$\mathsf{dim}\,\mathbf{D}^{\mathrm{b}}(X\text{-coh}) < \infty$$

Characterization of regular rings and schemes

The dimension of the category of perfect complexes over a ring or a scheme provides us a characterization of regular rings and schemes as follows:

- Let A be a noetherian ring. A is regular if and only if $\mathsf{dim}\,(A\text{-perf})) < \infty$.
- Let X be a quasi-projective scheme over a field k. X is regular if and only if $\mathsf{dim}\,(X\text{-perf})) < \infty$.

3.2 Some further applications

Due to the vastness of this topic and specific applications to different fields, I will only outline briefly two of the applications or related topics that come out from the study of this dimension function.

Finding algebras of representation dimension higher than 3

Auslander defined the *representation dimension* of an algebra as the minimal of the global dimension of the endomorphism groups of all generator-cogenerators. An intuitive way of seeing this dimension is, in words of Rouquier in [8]:

It was meant to measure how far an algebra is to having only finitely many classes of indecomposable modules.

By studying the dimension of the stable category (which is a triangulated category) of finitely generated modules of a self-injective algebra of finite dimension over a field k, Rouquier was able to find for the first time an example of a lower bound for the representation dimension. By taking the exterior algebra of the algebra k^n, one obtains that for $n \geq 3$, the representation dimension is greater than 4. This way, Rouquier provided the first known examples of algebras of representation dimension higher than 3.

Orlov spectrum

Orlov spectrum is defined as the set of integers formed by the different minimal numbers of steps required for the different objects of the category to generate it. Thus, Rouquier dimension is the minimum of Orlov's spectrum. When studying Orlov's spectrum, one can find gaps in the collection of integers. It turns out that these gaps are an invariant of the derived category of coherent sheaves and they encode some motivic and monodromy information about the category.

References

[1] Matthew Ballard, David Favero, and Ludmil Katzarkov. Orlov spectra: bounds and gaps. *Inventiones Mathematicae*, 189(2):359–430, 2012.

[2] Alexander Beĭlinson. Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra. 12(3):214–216, 1978.

[3] Alexei Bondal and Michel Van den Bergh. Generators and representability of functors in commutative and noncommutative geometry. 3:1–36, 2003.

[4] Henning Krause and Dirk Kussin. *Rouquier's theorem on representation dimension*, pages 95–103. Number 406 in Contemporary Mathematics. American Mathematical Society, 2006.

[5] Jon Peter May. The axioms for triangulated categories.

[6] Amnon Neeman. *Triangulated categories*. Number 148 in Annals of Mathematics Studies. Princeton University Press, 2001.

[7] Dmitri Orlov. Remarks on generators and dimensions of triangulated categories. *Moscow Mathematical Journal*, 9(1):513–519, 2009.

[8] Raphaël Rouquier. Representation dimension of exterior algebras. *Inventiones Mathematicae*, 165:357–367, 2006.

[9] Raphaël Rouquier. Dimensions of triangulated categories. *Journal of K-Theory*, 1:1–36, 2008.

[10] Jean-Louis Verdier. *Des catégories dérivées des catégories abéliennes*. PhD thesis, 1967.