ANAGRAMS: Grothendieck duality

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Abstract

These are the notes I prepared for my lectures at the ANAGRAMS seminar, January 2014. The goal was to introduce people to Grothendieck duality, by taking the scenic route from Riemann–Roch and Serre duality (with applications) to a description of the different proofs of Grothendieck duality.

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Lecture 1

Riemann–Roch theorem and Serre duality

Abstract

These are the notes for my first lecture on Grothendieck duality in the ANAGRAMS seminar. They discuss the Riemann–Roch theorem and Serre duality. A nice proof of Riemann–Roch is discussed. The applications of Riemann–Roch and Serre duality are given in the second lecture.

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1.1 Riemann–Roch

1.1.1 History

The Riemann–Roch theorem has already celebrated, or will almost celebrate its 150th birthday. In 1857 Bernhard Riemann proved *Riemann's inequality* (what this means will become clear later on) [5], while his student Gustav Roch found the missing term in 1865 [6], in order to make it an equality. Originally it was a theorem for Riemann surfaces, firmly rooted in complex analysis, and what is dubbed "the classical language" of divisors. With the advent of algebraic geometry the quest for an analogous statement began, and in 1931 Friedrich Karl Schmidt proved it for algebraic curves over perfect fields [7].

One can also look for Riemann–Roch-like statements for higher-dimensional, or singular objects. A version for smooth surfaces was proved by Guido Castelnuovo in 1896 (building on work of Max Noether from 1886 and Federigo Enriques[1] in 1894).

With the advent of sheaf theory and its use in algebraic geometry, Friedrich Hirzebruch proved in 1954 a version of Riemann–Roch (now oten dubbed Hirzebruch–Riemann–Roch) for compact complex manifolds of arbitrary dimensions. And in 1957 Alexander Grothendieck proved a far-reaching generalisation in the language of modern algebraic geometry: it is a relative statement, which has the previous results as a "trivial" case. This version is known as Grothendieck–Riemann–Roch, or its abbreviation GRR.

But in this seminar we will highlight a different route of generalisations. Whereas the previous generalisations were about proving certain numbers related to topological information to be equal, one can also consider the intrinsic geometric structure of objects, and look for relationships between associated structures. This is the result of Serre duality, obtained by Jean–Pierre Serre in 1955 [8]. It relates sheaf cohomology groups on non-singular projective algebraic varieties. It is related via Hodge theory to the maybe more familiar Poincaré duality if the base field is the complex numbers, but it considers the variety as a complex variety, not as a real manifold. One can obtain Riemann–Roch-like results this way, because it effectively reduces the amount of "abstract" cohomological information.

The final goal of this seminar is Grothendieck duality. This is a relative version of Serre duality, with a first proof by Robin Hartshorne in 1966 [3]. This proof is based on notes by Alexander Grothendieck, who envisioned the result in 1957 [1], but at the time the language required for the statement wasn't available. With the conception of derived categories [12], due to Jean–Louis Verdier[2] the generalisation become feasible.

It is still an area of active research, with many relations to other results. In the past 2 decades it has seen interesting new proofs and generalisations. As far as I understand one can obtain the results in the other branch of generalisations (i.e. the Hirzebruch–Riemann–Roch and Grothendieck–Riemann–Roch type of results), but this now requires a difficult argument based on the Lefschetz–Verdier formula [SGA5, exposé III].

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[1] Who apparently studied at the University of Liège.

[2] Its precise date is hard to pin down, he defended his PhD in 1967 but derived categories had been used for a few years by then.

The goal of this seminar series is to first discuss the classical results of Riemann–Roch and Serre duality. Then we go on to discuss some applications of these results, and start working towards Grothendieck duality in the second lecture. In the third lecture we will discuss Hartshorne's proof [3], while the fourth seminar is dedicated to a more modern proof by Daniel Murfet from 2007 [4].

1.1.2 Preliminaries

From now on I will use Ravi Vakil's notes on Riemann–Roch and Serre duality [10]. You can also take a look at [11] for a broader picture.

We first need to figure out what H^0 and H^1 are, in as concrete terms as possible. From now on we take C a nonsingular projective algebraic curve over an algebraically closed field k.

Global sections

Definition 1. Let $\mathcal{F}$ be a sheaf on C. Then $\mathrm{H}^0(C,\mathcal{F})$ are the *global sections* of $\mathcal{F}$ over C.

Example 2. Let $C = \mathbb{P}^1_k$. Then the global sections of $\mathcal{O}_{\mathbb{P}^1_k}$ on $\mathbb{P}^1_k$ are the constant functions, i.e.

$$\mathrm{H}^0(\mathbb{P}^1_k, \mathcal{O}_{\mathcal{P}^1_k}) = k. \tag{1.1}$$

To see this, observe that $\mathbb{P}^1_k$ is a gluing of two $\mathbb{A}^1_k$'s. The regular functions on one part $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ are the polynomials. But if we take a polynomial $f(x)$, the gluing procedure tells us that $f(1/x)$ should be a polynomial on the other $\mathbb{A}^1_k$, which is only possible if it is a constant. This is an algebraic analogue of *Liouville's theorem* in complex analysis.

We observe that the global sections have the structure of a k-vectorspace. This is the case for all $\mathcal{O}_C$-modules. In this case we define

$$\mathrm{h}^0(C,\mathcal{F}) := \dim_k \mathrm{H}^0(C,\mathcal{F}). \tag{1.2}$$

H^1 of a sheaf

Definition 3. Let $\mathcal{F}$ be an $\mathcal{O}_C$-module. Let $\mathfrak{U} = \{U_1, \ldots, U_n\}$ be an open cover of C. Denote $U_{i,j} = U_i \cap U_j$ and $U_{i,j,k} = U_i \cap U_j \cap U_k$. Then $\mathrm{H}^1(C,\mathcal{F})$ *as a set* consists of those tuples $(f_{i,j})_{i,j}$ where $f_{i,j} \in \mathrm{H}^0(U_{i,j},\mathcal{F})$ such that $f_{i,j} - f_{j,k} + f_{i,k} = 0$ in $\mathrm{H}^0(U_{i,j,k},\mathcal{F})$. We will call these *cocycles*.

We consider $\mathrm{H}^1(C,\mathcal{F})$ *as an abelian group* by declaring a tuple $(f_{i,j})_{i,j}$ zero if there are sections $g_i \in \mathrm{H}^0(U_i,\mathcal{F})$ such that $f_{i,j} = g_i - g_j$ in $\mathrm{H}^0(U_{i,j},\mathcal{F})$. And we get $\mathrm{H}^1(C,\mathcal{F})$ *as a k-vectorspace* by the k-vectorspace structure on $\mathrm{H}^0(C,\mathcal{F})$.

The definition of the zero in this vectorspace explains what H^1 is about: it measures to which extent we cannot glue global sections.

Of course, this was for a fixed covering. There is a partial order on coverings, and if we have a cocycle on $\mathfrak{U}$, with $\mathfrak{U} \leq \mathfrak{V}$, then by restricting it we get a cocycle on the finer covering $\mathfrak{V}$. So to be strict we have to take the direct limit over these coverings. To get some more background, see [2, exercises III.4.4 and III.4.11].

Example 4. Let G be an abelian group, and denote $\underline{G}$ the associated constant sheaf on C. Then $\mathrm{H}^1(C,\underline{G}) = 0$, as there are no obstructions whatsoever to glue sections.

Example 5. Take $p \in C$ and define a *skyscraper sheaf* k_p on C by

$$(1.3)\quad \Gamma(U, k_p) = \begin{cases} k & p \in U \\ 0 & p \notin U \end{cases}$$

for $U \subseteq C$ open, which is an $\mathcal{O}_C$-module by the obvious multiplication. As in the previous case we get $\mathrm{H}^1(C, k_p) = 0$.

These are both examples of *flasque sheaves*, and these never have higher cohomology.

The Euler characteristic Again the cohomology groups have a k-vectorspace structure, which allows us to define

$$(1.4)\quad \mathrm{h}^1(C, \mathcal{F}) := \dim_k \mathrm{H}^1(C, \mathcal{F})$$

and in general we will set

$$(1.5)\quad \mathrm{h}^i(C, \mathcal{F}) := \dim_k \mathrm{H}^i(C, \mathcal{F}).$$

By a nice result of Grothendieck we know that sheaf cohomology vanishes above the dimension of the variety [2, theorem III.2.7]. Hence in the case of a curve there is only a H^0 and a H^1. We then define the *Euler characteristic*

$$(1.6)\quad \chi(C, \mathcal{F}) := \mathrm{h}^0(C, \mathcal{F}) - \mathrm{h}^1(C, \mathcal{F}).$$

In general this will be an alternating sum over more terms, up to the dimension of the variety. We will use this definition later on, the philosophy of "taking together all the cohomology groups" proves to be very fruitful.

A short exact sequence of sheaves Take $p \in C$. Then $\mathcal{O}_C(-p)$ is the sheaf of regular functions with a zero in p. Then we have a short exact sequence

$$(1.7)\quad 0 \to \mathcal{O}_C(-p) \to \mathcal{O}_C \to k_p \to 0$$

where the first morphism is the obvious inclusion, and the second morphism is taking the value at p. Moreover, for any invertible sheaf $\mathcal{L}$ we have a short exact sequence

$$(1.8)\quad 0 \to \mathcal{L}(-p) \to \mathcal{L} \to k_p \to 0$$

by taking the sections of $\mathcal{L}$ which vanish at p.
The associated long exact sequence in cohomology yields

(1.9)
$$0 \to \mathrm{H}^0(C, \mathcal{L}(-p)) \to \mathrm{H}^0(C, \mathcal{L}) \to \mathrm{H}^0(C, k_p) \to \mathrm{H}^1(C, \mathcal{L}(-p)) \to \mathrm{H}^1(C, \mathcal{L}) \to 0$$

hence the Euler characteristic is additive:

$$(1.10)\quad \chi(C, \mathcal{L}(-p)) = \chi(C, \mathcal{L}) - \chi(C, k_p) = \chi(C, \mathcal{L}) - 1.$$

This will be used in the proof of (cheap) Riemann–Roch, see lemma 8.

1.1.3 Statement: curves

We can now give a first version of the Riemann–Roch theorem. The statement requires the canonical sheaf Ω^1_C, or line bundle of differentials, which will be introduced later.

The line bundle of differentials on a curve is the source of an important invariant of the topology of the curve.

Definition 6. The *genus* of a curve is $g_C := \dim_k \mathrm{H}^0(C, \Omega^1_C)$.

Theorem 7 (Riemann–Roch)**.** Let $\mathcal{L}$ be an invertible sheaf of degree d on C. Let g be the genus of C Then

(1.11) $\mathrm{h}^0(C,\mathcal{L}) - \mathrm{h}^0(C,\Omega^1_C \otimes \mathcal{L}^\vee) = d - g + 1,$

where $\mathcal{L}^\vee$ is the dual of $\mathcal{L}$, given by $\mathcal{H}\mathrm{om}(\mathcal{L},\mathcal{O}_C)$.

Hence Riemann–Roch is a relationship between some numbers: if we know all but one of them we know all of them. The number we care most about is $\mathrm{h}^0(C,\mathcal{L})$. When we take $\mathcal{L} = \mathcal{O}_C(D)$ we are interested in the dimension of the space of functions with "prescribed behaviour at D": we require the poles to be no worse than what is allowed by D. As for most points the coefficient of $p \in C$ will be zero, a section of $\mathcal{O}_C(D)$ has (possibly) some poles in the points of D and (possibly) zeroes of at least a certain order.

As hinted at in the introduction, we consider the second term of the left-hand side as a "correction term". So in general we get

(1.12) $\mathrm{h}^0(C,\mathcal{L}) \geq d - g + 1$

which is called *Riemann's equality*, and an important question is whether we have equality in certain cases.

The following is dubbed "cheap Riemann–Roch" by Ravi Vakil: it is a first step in proving Riemann–Roch and Serre duality for curves. Because it uses both H^0 and H^1 it is "cheap": H^1 is hard to understand. But it shows that

1. Riemann–Roch is about giving a numerical relationship: if D and D' are divisors such that $\deg(D) = \deg(D')$ they behave similarly;
2. one should consider all the sheaf cohomology groups together: the Euler characteristic is well-behaved, separate dimensions are not.

Lemma 8 (Cheap Riemann–Roch)**.** We have

(1.13) $\chi(C,\mathcal{L}) = \deg(\mathcal{L}) + \chi(C,\mathcal{O}_C),$

hence for $\mathcal{L} = \mathcal{O}_C(D)$ we get

(1.14) $\chi(C,\mathcal{O}_C(D)) = \mathrm{h}^0(C,\mathcal{O}_C(D)) - \mathrm{h}^1(C,\mathcal{O}_C(D)) = d + 1 - \mathrm{h}^1(C,\mathcal{O}_C).$

Proof. The invertible sheaf $\mathcal{L}$ can always be written as $\mathcal{O}_C(p_1+\ldots+p_a-q_1-\ldots-q_b)$ for p_i and q_j points on C, because in this case the Picard group (the group of invertible sheaves) is isomorphic to the group of Cartier divisors (which agree with Weil divisors, the most down-to-earth version we use here). We get $a - b = \deg(\mathcal{L})$.

Then we can set up an induction on the number of points $a+b$: for $\mathcal{O}_C$ it is obvious, and adding a point uses the additivity of the Euler characteristic on short exact sequences. $\square$

1.1.4 Statement: surfaces

Theorem 9 (Riemann–Roch for surfaces)**.** Let $\mathcal{L}$ be an invertible sheaf on a smooth projective algebraic surface S. Then

$$
\begin{aligned}
\chi(S,\mathcal{L}) &= \mathrm{h}^0(S,\mathcal{L}) - \mathrm{h}^1(S,\mathcal{L}) + \mathrm{h}^2(S,\mathcal{L}) \\
&= \chi(S,\mathcal{O}_S) + \frac{1}{2}(\mathcal{L}^2 - \mathcal{L}\cdot\omega_S) \\
&= \mathrm{h}^0(S,\mathcal{O}_S) - \mathrm{h}^1(S,\mathcal{O}_S) + \mathrm{h}^2(S,\mathcal{O}_S) + \frac{1}{2}(\mathcal{L}^2 - \mathcal{L}\cdot\omega_S)
\end{aligned}
\tag{1.15}
$$

where we take the intersection numbers of the divisors associated to $\mathcal{L}$ and ω_S. Moreover we have *Noether's formula*

$$\chi(S,\mathcal{O}_S) = \frac{\omega_S\cdot\omega_S + \chi_{\mathrm{top}}(S)}{12} \tag{1.16}$$

and if C is a curve on S we have the *genus formula*

$$2\mathrm{p_a}(C) - 2 = C^2 + C\cdot \mathrm{K}_X. \tag{1.17}$$

Again this is a (or rather, its different manifestations are) numerical relation(s) on numbers associated to the surface and an invertible sheaf (or divisor) on it. We have a H^2 popping up, which we don't understand at the moment. But once we've seen Serre duality we will know how to reduce this H^2 to a H^0. Then we get a formula containing H^0 and H^1.

This H^1-term wasn't known at first (we are now in the era of the Italian school) and hence we only had an inequality. The failure of this equality was called the *superabundance*. In the second lecture we will discuss some other interesting facts about this.

1.2 Serre duality

1.2.1 Statement

In the statement of Riemann–Roch we used the canonical sheaf ω_X (or Ω^1_C for curves, as they agree in dimension one), which made the magic work. In the more general setting that we will enter now (possible singularities) we will need a more general object serving the role of the canonical sheaf. We will define its properties, and then we are left with an existence question. This exposition is taken from [2, §III.7], in later seminars we will give more general statements and come back to this setup.

Definition 10. Let X/k be a proper n-dimensional variety. A *dualising sheaf* for X is a coherent sheaf ω_X° together with a *trace morphism* $\mathrm{tr}\colon \mathrm{H}^n(X,\omega_X^\circ)\to k$, such that for all $\mathcal{F}\in\mathrm{Coh}/X$ the natural pairing

$$\mathrm{Hom}(\mathcal{F},\omega_X^\circ)\times\mathrm{H}^n(X,\mathcal{F})\to\mathrm{H}^n(X,\omega_X^\circ) \tag{1.18}$$

composed with tr gives an isomorphism

$$\mathrm{Hom}(\mathcal{F},\omega_X^\circ)\cong\mathrm{H}^n(X,\mathcal{F})^\vee. \tag{1.19}$$

So the first part of duality theory concerns the *existence of this dualising sheaf*. A sufficient condition is that X is projective [2, proposition III.7.5]. In later seminars more general existence conditions will be discussed. One can prove that such a dualising sheaf and trace morphism are unique if they exist [2, proposition III.7.2].

Example 11. The most trivial case one can image is $X = \operatorname{Spec} k$ a point. Then coherent sheaves are finite-dimensional vectorspaces, and $\omega_X^\circ = k$. The isomorphism tr is then the definition of the dual vectorspace.

The next part concerns the *actual duality*. We state [2, proposition III.7.6].

Theorem 12 (Serre duality)**.** Let X/k be a projective n-dimensional variety. Let ω_X° be its dualising sheaf. Then for all $i \geq 0$ and $\mathcal{F} \in \operatorname{Coh}/X$ we have functorial maps

$$\theta^i \colon \operatorname{Ext}^i(\mathcal{F}, \omega_X^\circ) \to \mathrm{H}^{n-i}(X, \mathcal{F})^\vee \tag{1.20}$$

such that θ^0 corresponds to tr. Moreover, if X is Cohen–Macaulay[3] the θ^i are isomorphisms for all $i \geq 0$ and $\mathcal{F} \in \operatorname{Coh}/X$.

The following corollary illustrates nicely why Serre duality is truly a duality result: it gives a relationship between H^i and H^{n-i}.

Corollary 13. Let X be projective Cohen–Macaulay of (equi-)dimension n over k. Let $\mathcal{F}$ be a locally free sheaf on X. Then we have isomorphisms

$$\mathrm{H}^i(X, \mathcal{F}) \cong \mathrm{H}^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ)^\vee. \tag{1.21}$$

1.2.2 Proof of the curve case

A version of Serre duality for curves states the following.

Theorem 14. There is a natural perfect pairing

$$\mathrm{H}^0(C, \Omega_C^1(-D)) \times \mathrm{H}^1(C, \mathcal{O}_C(D)) \to k. \tag{1.22}$$

If we assume this theorem for now, the proof of the Riemann–Roch theorem becomes an easy corollary.

Proof of Riemann–Roch using Serre duality. We have that

$$\begin{aligned}
&\mathrm{h}^0(C, \mathcal{L}) - \mathrm{h}^0(C, \Omega^1 \otimes \mathcal{L}^\vee) && \\
&\quad = \mathrm{h}^0(C, \mathcal{L}) - \mathrm{h}^1(C, \mathcal{L}) && \text{Serre duality} \\
&\quad = \chi(C, \mathcal{L}) && \text{definition of } \chi \\
&\quad = d + \chi(C, \mathcal{O}_C) && \text{cheap Riemann–Roch} \\
&\quad = d + \mathrm{h}^0(C, \mathcal{O}_C) - \mathrm{h}^1(C, \mathcal{O}_C) && \text{definition of } \chi \\
&\quad = d + 1 - \mathrm{h}^1(C, \mathcal{O}_C) && \text{global sections are constants} \\
&\quad = d + 1 - \mathrm{h}^0(C, \Omega_C^1) && \text{Serre duality} \\
&\quad = d + 1 - g && \text{definition of } g.
\end{aligned} \tag{1.23}$$

□

[3] A technical condition that says that "mild singularities" are allowed. It means that each local ring has Krull dimension equal to the depth (we always have that depth is bounded above by Krull dimension), where depth corresponds to the length of a maximal regular sequence for the local ring itself. One can just read non-singular, which is the case we will need in later applications.

The rest of this section is dedicated to the proof of Serre duality in the curve case. It is taken from Vakil's notes, which are based on [9, §2] and originate from a proof by Weil. The original text of Serre is (as usual) beautiful, and definitely deserves a reading. It's hard to believe it was written in the fifties.

Adèles When I was preparing these notes this part scared me, because "adèle" is a scary word used by people who know something about class field theory[4]. I am not one of them. The approach of the proof of Riemann–Roch taken by Vakil, Serre and Weil is by considering "pre-adèles" or repartitions. This avoids the technical machinery of class field theory (it would be insane to use it to prove something as down-to-earth as Riemann–Roch) and has a nice interpretation in terms of the geometry. By preparing these notes I finally got myself familiar with adèles, so I hope other people will benefit too from advertising this approach.

Before we start by building things from the ground up, remark that the occurence of techniques from class field theory is not too far-fetched. It deals with fields, there is a bijection between curves and their function fields [2, §I.6], and the ring of adèles of the function field of a curve satisfies self-duality which is one of the results in Tate's thesis. This self-duality implies Riemann–Roch, and we will develop as much of class field theory as required for the proof. So let's get started.

The part on $\mathrm{I}(D)$ To stick to Serre's notation we will denote

$$\mathrm{I}(D) := \mathrm{H}^1(C, \mathcal{O}_C(D)). \tag{1.24}$$

The dimension of this vectorspace pops up in the statement of cheap Riemann–Roch, and we decided that h^1 is not an easy invariant. Hence we would like to get a better understanding.

Definition 15. A *repartition* is an indexed set $(f_p)_{p\in C}$ with $f_p \in k(C)$ for all $p \in C$ such that $f_p \in \mathcal{O}_{C,p}$ for all but finitely many points p. The set R of repartitions comes equipped with a ring structure (pointwise addition and multiplication), with $k(C)$ being a subring of this (if $f \in k(C)$ we take $f_p = f$, which is regular at all but finitely many points of C), and R being a $k(C)$-algebra.

Hence a repartition is a collection of rational functions, indexed by the points of the curve, such that at most finitely many rational functions have a pole in the point at which they are associated. This ring contains tons of potential information (recall that H^1 was about gluing local sections to a global section, and the obstructions in doing so), and we wish to put it to good use.

Definition 16. Let D be a divisor on C. We set

$$\mathrm{R}(D) := \{(f_p)_{p\in C} \mid \mathrm{v}_p(f_p) + \mathrm{v}_p(D) \geq 0\}, \tag{1.25}$$

an additive subgroup of R.

This is analogous to $\mathcal{O}_C(D)$, but taken for each point separately. Recall that $\mathcal{O}_C(D)$ is the sheaf of meromorphic functions on C with prescribed behaviour in D: if n_p is the coefficient of the point p in D, then we require at most a pole of order n_p is n_p is positive, or at least a zero of order n_p if n_p is negative[5].

We can now interpret H^1 in terms of these objects.

[4] It is also the street in which the math department of Université Paris-Sud is located.

[5] Sometimes the other convention is used...

Proposition 17. We have

(1.26) $\mathrm{I}(D) = \mathrm{H}^1(C, \mathcal{O}_C(D)) \cong R/(\mathrm{R}(D) + k(C))$.

Proof. Associated to the field $k(C)$ we have the constant sheaf $\underline{k(C)}$ on C. We have a natural injection of $\mathcal{O}_C(D)$ into this constant sheaf, and we define S to be the cokernel of this injection, i.e. we have the short exact sequence

(1.27) $0 \to \mathcal{O}_C(D) \to \underline{k(C)} \to S \to 0$.

Taking global sections we get

(1.28) $\ldots \to k(C) \to \mathrm{H}^0(C, S) \to \mathrm{H}^1(C, \mathcal{O}_C(D)) \to \mathrm{H}^1(C, \underline{k(C)}) = 0$

because constant sheaves don't have higher cohomology groups. Hence we have to prove that

(1.29) $\mathrm{H}^0(C, S) \cong R/\mathrm{R}(D)$.

To prove this, we have to interpret S as the quotient sheaf of $\underline{k(C)}$, which we do by looking at its stalks. If p is a point of C, we have

(1.30) $S_p = (k(C)/\mathcal{O}_C(D))_p = \{f \in k(C) \mid \mathrm{v}_p(f) \geq -\mathrm{v}_p(D)\}$.

Hence

(1.31) $R/\mathrm{R}(D) = \bigoplus_{p \in C} S_p$,

the quotient is a sum of skyscraper sheaves, i.e. a again a skyscraper sheaf.

We wish to show that S equals this same direct sum of skyscraper sheaves, i.e. that sections of S consist of a selection of values of S_p for all p, almost all of which are zero. Elements of the stalk are represented by giving an open neighbourhood of the point and a section on this neighbourhood, and sections that are equal on some smaller neighbourhood are identified.

So let p be a point of C, and let $s \in S(U)$ be a section defined on an open neighbourhood U of p. We wish to show that it is a section of the sum of skyscraper sheaves. To do so we look for a (smaller) neighbourhood $U' \subseteq U$ of p such that $s|_{U' \setminus \{p\}} = 0$, because then s belongs to the skyscraper sheaf associated to p.

It suffices to take this smaller neighbourhood disjoint from

- $\mathrm{supp}(D) \setminus \{p\}$ (i.e. we ignore the points of the divisor, except potentially p because we are interested in this point),
- $s^{-1}(\infty) \setminus \{p\}$ (i.e. away from the poles of s, except for p of course because we wish to include the point in our neighbourhood).

In order to use this setup: take s an element of the stalk S_p. It has a lift in some neighbourhood to a section s' of the constant sheaf $\underline{k(C)}$ (because S is defined as a quotient sheaf). On the U' (relative to the neighbourhood of p we used to obtain the lift) everything is regular, so we can choose a section of $\mathcal{O}_C(D)$ that cancels what is going on for s' on U' and in the quotient for the stalk it becomes zero when restricted to $U' \setminus \{p\}$. ☐

The part on $\mathrm{J}(D)$ and J We now set

(1.32) $\mathrm{J}(D) := \mathrm{I}(D)^\vee = (R/(\mathrm{R}(D) + k(C)))^\vee$.

Hence an element of $\mathrm{J}(D)$ is a k-linear form on R (our huge ring of repartitions) which vanishes on $\mathrm{R}(D)$ and $k(C)$. Because $D \leq D'$ implies $\mathrm{R}(D) \subseteq \mathrm{R}(D')$, hence $\mathrm{J}(D') \subseteq \mathrm{J}(D)$ we can define

(1.33) $J := \bigcup_D \mathrm{J}(D)$.

Lemma 18. *J is a $k(C)$-vectorspace.*

Proof. Take $f \in k(C)$ and $\alpha \in J$. We consider

(1.34) $f\alpha \colon R \to k : r \mapsto \langle \alpha, fr \rangle$

which is a linear functional on R, which vanishes on $k(C)$. This assignment gives J the structure of a $k(C)$-vectorspace: take $\alpha \in \mathrm{J}(D)$ and $(f) = D'$. Then the linear form $f\alpha$ vanishes on $\mathrm{R}(D - D')$, because if $r \in \mathrm{R}(D - D')$ then $fr \in \mathrm{R}(D)$, hence $\langle \alpha, fr \rangle = 0$. So $f\alpha$ belongs to $\mathrm{J}(D - D')$, and therefore to J. □

Moreover, whereas it is a horribly big k-vectorspace, it is well-behaved as a $k(C)$-vectorspace.

Lemma 19. We have $\dim_{k(C)} J \leq 1$.

Proof. Take α, β linearly independent over $k(C)$. We can find a divisor D such that $\alpha, \beta \in \mathrm{J}(D)$, and denote $d = \deg(D)$.

Assume that D_n is any divisor such that $\deg(D_n) = n$. Then for each section $f, g \in \mathrm{H}^0(C, \mathcal{O}_C(D_n))$ we get that $f\alpha \in \mathrm{J}(D - D_n)$ by the previous argument, and similarly $g\beta \in \mathrm{J}(D - D_n)$. Because α and β are linearly independent we know that

$$\begin{aligned} \mathrm{H}^0(C, \mathcal{O}_C(D_n)) \oplus \mathrm{H}^0(C, \mathcal{O}_C(D_n)) &\to \mathrm{J}(D - D_n) \\ (f, g) &\mapsto f\alpha + g\beta \end{aligned} \tag{1.35}$$

is an injection, hence

(1.36) $\dim_k \mathrm{J}(D - D_n) \geq 2\,\mathrm{h}^0(C, \mathcal{O}_C(D_n))$.

On the left-hand side we have

$$\begin{aligned} &\dim_k \mathrm{J}(D - D_n) \\ &\quad = \dim_k \mathrm{I}(D - D_n) && \text{definition} \\ &\quad = \mathrm{h}^1(C, \mathcal{O}_C(D - D_n)) && \text{proposition 17} \\ &\quad = \mathrm{h}^0(C, \mathcal{O}_C(D - D_n)) - (d - n) + \text{constant} && \text{cheap Riemann–Roch} \\ &\quad = n + \text{constant}' && n \gg 0. \end{aligned} \tag{1.37}$$

In this case, constant means independent of n or D_n.

On the right-hand side we get by cheap Riemann–Roch (lemma 8) that

(1.38) $2\,\mathrm{h}^0(C, \mathcal{O}_C(D_n)) \geq 2\deg(D_n) + \text{constant}''$

so if $n \gg 0$ we get a contradiction as two sides cannot be equal. This dimension count over k proves that α and β cannot be linearly independent, which proves that $\dim_{k(C)} J \leq 1$. □

The part on differentials We will need to know what differentials on a curve are. In courses on differential geometry or complex analysis one has seen these before. In the algebraic geometry case one uses Kähler differentials to have a nice analogue. It provides another natural sheaf, besides the structure sheaf (and its twists). On a curve it will be an invertible sheaf, denoted Ω^1_C and it has an associated canonical divisor K_C.

Another piece of notation that we will use is the set of meromorphic differentials M, which is a one-dimensional $k(C)$-vectorspace as these are exactly the objects that (locally) look like $f(z)\mathrm{d}z$, with $f \in k(C)$. As this is a one-dimensional vectorspace, we just take the divisor associated to any meromorphic differential, and this will be the canonical divisor introduced before.

We can now consider some differential forms (both meromorphic and holomorphic) on a curve. The terminology is mildly inspired by complex geometry here.

Example 20. Take $C = \mathbb{P}^1_k$. Then we have a differential form $\omega = \mathrm{d}z$ on $\mathbb{A}^1_k$, and as the transition map to the second chart is $z \mapsto z^{-1}$ we get that $\mathrm{d}(z^{-1}) = -z^{-2}\mathrm{d}z$, hence this differential form has a pole of order two at ∞. Its associated canonical divisor is -2∞. There are no holomorphic differentials on $\mathbb{P}^k_1$, its genus is zero.

Example 21. Take $C = \mathbb{C}/\Lambda$ a complex elliptic curve, defined by taking a quotient by a torus. There is a differential form $\omega = \mathrm{d}z$ on $\mathbb{C}$, which induces a differential form on C. But as it is everywhere holomorphic it has no poles (nor zeroes), and the canonical divisor is zero. Hence the only holomorphic differential forms are the constants, and its genus is one.

For every point $p \in C$ there is a *residue map*

(1.39) $\mathrm{res}_p \colon M \to k,$

analogous to the case of complex analysis. To define this map we can write a meromorphic differential locally around p as

(1.40) $(a_{-n}/t^n + \ldots + a_{-1}/t)\mathrm{d}t + \text{regular part}$

where t is a uniformising parameter, and set the residue equal to a_{-1}. It requires some work to prove that this is independent of the choice of local parameter (especially in the case of positive characteristic) [9][6].

We moreover have the *residue theorem*, which says that for a meromorphic differential $\omega \in M$ we have

(1.41) $\displaystyle\sum_{p \in C} \mathrm{res}_p(\omega) = 0.$

The complex analytic case can be done by Stokes' theorem, for the general case we refer to [9].

The setup for the final part of the proof We wish to show that two vectorspaces are dual to eachother, and we will explicitly construct a linear functional for this.

[6]As suggested in Ravi Vakil's notes, if one assumes the complex analytic case, there is a nice proof for the general case, as suggested by Kiran Kedlaya. Uniformising parameters t and u are related by $t = u + \sum_{k=2}^{+\infty} c_k u^k$, so if $\sum_{i=-n}^{-1} a_i t^i \mathrm{d}t$ and $\sum_{i=-n}^{-1} b_i u^i \mathrm{d}u$ are two local expressions, we get a polynomial identity in the a_i's and b_i's whose coefficients are integers. But over $\mathbb{C}$ we have an equality, hence the polynomial identity reduces to an identity over every field.

To do so, take $\omega \in M$ a meromorphic differential on C. We define the divisor

$$(\omega) := \sum_{p \in C} \mathrm{v}_p(\omega) p \tag{1.42}$$

hence the sheaf $\Omega^1(-D)$ is the sheaf of differentials such that $(\omega) \leq D$.
Then we define the pairing

$$\langle -, - \rangle : M \times R \to k : (\omega, r) \mapsto \langle \omega, r \rangle = \sum_{p \in C} \mathrm{res}_p(r_p \omega). \tag{1.43}$$

It has the following properties.

Lemma 22. The pairing $\langle -, - \rangle$ satisfies

1. $\langle \omega, r \rangle = 0$ if $r \in k(C)$;
2. $\langle \omega, r \rangle = 0$ if $r \in \mathrm{R}(D)$ and $\omega \in \mathrm{H}^0(C, \Omega^1_C(-D))$;
3. $\langle f\omega, r \rangle = \langle \omega, fr \rangle$ if $f \in k(C)$.

Proof.
1. This is the residue theorem.
2. The product $r_p \omega$ cannot have a pole, for any $p \in C$, because the zeroes must at least cancel the poles by the assumptions on r and ω.
3. Both pairings evaluate to a sum of residues over $f\omega r$.

□

For each meromorphic differential ω in $\mathrm{H}^0(C, \Omega^1_C(-D))$ we have a linear functional $\theta(\omega)$ on R, and by items 1 and 2 of lemma 22 it is also a linear functional on $R/(\mathrm{R}(D) + k(C))$. Hence we get a map

$$\theta : \mathrm{H}^0(C, \Omega^1_C(-D)) \to \mathrm{J}(D) \tag{1.44}$$

as $\mathrm{J}(D)$ is shorthand for the dual of $R/(\mathrm{R}(D) + k(C))$ by proposition 17. This θ is moreover defined as a map $M \to J$ in general. But we have the following nice property, that relates the more general map to the specific map.

Lemma 23. Let ω be a meromorphic differential such that $\theta(\omega) \in \mathrm{J}(D)$. Then we have that $\omega \in \mathrm{H}^0(C, \Omega^1_C(-D))$[7].

Proof. Assume on the contrary that $\omega \notin \Omega^1(-D)$. This means that there is a point $p \in C$ such that ω has a pole in p that is bigger than allowed by D, or symbolically

$$\mathrm{v}_p(\omega) < \mathrm{v}_p(-D). \tag{1.45}$$

Then we take a repartition $r \in \mathrm{R}(D)$ by setting

$$r_q = \begin{cases} 0 & q \neq p \\ 1/t^{\mathrm{v}_p(\omega)+1} & q = p. \end{cases} \tag{1.46}$$

[7] Remark that Serre denotes this sheaf $\Omega^1_C(D)$, because for differentials he reverses the terminology.

Because

(1.47) $\mathrm{v}_p(r_p\omega) = -1$

we get

(1.48) $\langle\omega, r\rangle = \sum_{q\in C} \mathrm{res}(r_q\omega) = \mathrm{res}(r_p\omega) \neq 0.$

But this means $\theta(\omega)$ is not zero on $\mathrm{R}(D)$, but this is required by the definition of θ, hence we obtain a contradiction. □

Recall that we wish to prove that θ induces an isomorphism from $\mathrm{H}^0(C, \Omega^1_C(-D))$ to $\mathrm{H}^1(C, \mathcal{O}_C(D))^\vee$, and this last object is also denoted $\mathrm{J}(D)$.

Proof of Serre duality for curves. To see that θ is *injective*, take $\omega \in \mathrm{H}^0(C, \Omega^1_C(-D))$ such that $\theta(\omega) = 0$. Then by lemma 23 we have that $\omega \in \Omega^1_C(-D')$ for every divisor D', which implies $\omega = 0$, as all possible configurations of poles and zeroes should be valid at the same time.

To see that θ is *surjective*, observe that by item 3 of lemma 22 we have that θ is $k(C)$-linear, from M to J. By definition we have $\dim_{k(C)} M = 1$, by lemma 19 we have $\dim_{k(C)} J \leq 1$. An injection of finite-dimensional vectorspaces into a smaller vectorspace is necessarily surjective.

Hence if α is an element of $\mathrm{J}(D)$ we get a meromorphic differential ω such that $\theta(\omega)\alpha$, and lemma 23 shows that $\omega \in \Omega(-D)$. □

1.2.3 The general case

The proof for curves has an explicit flavour in terms of residues to it. In higher dimensions we lose this. For a nice (but abstract) proof one can take a look at [2, §III.7]. The proof goes as follows, for $X \subseteq \mathbb{P}^n_k$:

1. Prove Serre duality for $\mathbb{P}^n_k$, which is very concrete (see later). The dualizing sheaf $\omega_{\mathbb{P}^n_k} = \bigwedge^n \Omega_{\mathbb{P}^n_k/k}$ is $\mathcal{O}_{\mathbb{P}^n_k}(-n-1)$.
2. Prove that $\omega_X^\circ = \mathcal{E}\mathrm{xt}^r_{\mathbb{P}^n_k}(\mathcal{O}_X, \omega_{\mathbb{P}^n_k})$ is a dualising sheaf for X, where r is the codimension of X. Remark that $\mathcal{E}\mathrm{xt}^i_{\mathbb{P}^n_k}(\mathcal{O}_X, \omega_{\mathbb{P}^n_k}) = 0$ for all $i < r$.

The question becomes: can we interpret this dualising sheaf? We know that for $\mathbb{P}^n_k$ that the dualising sheaf is given by the canonical sheaf, hence the abuse of notation. Similarly, we have that for X nonsingular projective that this is true. The statement of Serre duality holds for Cohen–Macaulay varieties, i.e. we allow mild singularities. But then it's harder to interpret the dualising sheaf.

For completeness' sake we can give the duality result for $\mathbb{P}^n_k = \mathrm{Proj}\, k[x_0, \ldots, x_n]$. Recall that $\mathcal{O}_{\mathbb{P}^n_k}(1)$ consists of the linear forms on $\mathbb{P}^n_k$, and higher twists corresponds to higher-degree equations.

Theorem 24 (Serre duality for $\mathbb{P}^n_k$)**.** We have that

1. $\mathrm{H}^i(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(r)) = 0$ for all $0 < i < n$ and $r \in \mathbb{Z}$.
2. $\mathrm{H}^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-n-1)) \cong k$;

$r \backslash i$	-1	0	1	2
3	0	4	0	0
2	0	3	0	0
1	0	2	0	0
0	0	1	0	0
-1	0	0	0	0
-2	0	0	1	0
-3	0	0	2	0
-4	0	0	3	0
-5	0	0	4	0

Table 1.1: $\mathrm{h}^i(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(r))$

$r \backslash i$	-1	0	1	2	3
4	0	15	0	0	0
3	0	10	0	0	0
2	0	6	0	0	0
1	0	3	0	0	0
0	0	1	0	0	0
-1	0	0	0	0	0
-2	0	0	0	0	0
-3	0	0	0	1	0
-4	0	0	0	3	0
-5	0	0	0	6	0
-6	0	0	0	10	0
-7	0	0	0	15	0

Table 1.2: $\mathrm{h}^i(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(r))$

3. $\mathrm{H}^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(r)) \times \mathrm{H}^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-n-r-1)) \to \mathrm{H}^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-n-1)) \cong k$ is a perfect pairing of k-vectorspaces.

This yields the pictures in tables 1.1 to 1.4.

r \ i	-1	0	1	2	3	4
5	0	56	0	0	0	0
4	0	35	0	0	0	0
3	0	20	0	0	0	0
2	0	10	0	0	0	0
1	0	4	0	0	0	0
0	0	1	0	0	0	0
-1	0	0	0	0	0	0
-2	0	0	0	0	0	0
-3	0	0	0	0	0	0
-4	0	0	0	0	1	0
-5	0	0	0	0	4	0
-6	0	0	0	0	10	0
-7	0	0	0	0	20	0
-8	0	0	0	0	35	0
-9	0	0	0	0	56	0

Table 1.3: $\mathrm{h}^i(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(r))$

r \ i	-1	0	1	2	3	4	5
6	0	210	0	0	0	0	0
5	0	126	0	0	0	0	0
4	0	70	0	0	0	0	0
3	0	35	0	0	0	0	0
2	0	15	0	0	0	0	0
1	0	5	0	0	0	0	0
0	0	1	0	0	0	0	0
-1	0	0	0	0	0	0	0
-2	0	0	0	0	0	0	0
-3	0	0	0	0	0	0	0
-4	0	0	0	0	0	0	0
-5	0	0	0	0	0	1	0
-6	0	0	0	0	0	5	0
-7	0	0	0	0	0	15	0
-8	0	0	0	0	0	35	0
-9	0	0	0	0	0	70	0
-10	0	0	0	0	0	126	0
-11	0	0	0	0	0	210	0

Table 1.4: $\mathrm{h}^i(\mathbb{P}^4_k, \mathcal{O}_{\mathbb{P}^4_k}(r))$

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Lecture 2

More on Riemann–Roch and Serre duality, with applications

Abstract

These are the notes for my second lecture on Grothendieck duality in the ANAGRAMS seminar. They continue the discussion of Riemann–Roch and Serre duality started in the first lecture, by giving some applications.

Contents

2.1 Applications of Riemann–Roch for curves

As the first lecture didn't allow for a discussion of any applications I have moved these to (the notes of) the second lecture.

2.1.1 Geometric genus equals arithmetic genus

For a projective nonsingular curve C we get that

$$\mathrm{p}_{\mathrm{arith}}(C) = 1 - \chi(C, \mathcal{O}_C) = \mathrm{h}^1(C, \mathcal{O}_C) = \mathrm{h}^0(C, \Omega^1_C) = \mathrm{p}_{\mathrm{geom}}(C). \tag{2.1}$$

Hence the arithmetic and geometric genus agree. The geometric genus "counts the number of holes" whereas the arithmetic genus doesn't really count anything (it can be negative in more general situations[1], because its generalisation is an alternating sum).

2.1.2 Curves of degree d in $\mathbb{P}^n_k$

If C is given as a curve of degree d in $\mathbb{P}^n_k$ we can take a hyperplane section $C \cap H = D$ for a divisor. Then we get

$$\chi(\mathcal{L}(D)) = d + 1 - \mathrm{p}_{\mathrm{a}}(C) \tag{2.3}$$

as the degree of the divisor D is always d, which can be considered an application of Bézout's theorem.

2.1.3 Vanishing of H^1

The theorem of Riemann–Roch should be view in terms of the *Riemann–Roch problem*, which is the study of the (asymptotic) behaviour of $\mathrm{h}^0(C, \mathcal{O}_C(np))$, for $p \in C$, or more general divisors. So the question becomes: can we determine h^1, or the h^0 of $\omega_C(-D)$? The answer is given in [1, remark IV.1.3.2]:

Lemma 25. If $\deg(D) > 0$ and $n \deg(D) > \deg(\mathrm{K}_C)$ then $\mathrm{h}^0(C, \mathcal{O}_C(\mathrm{K}_C - nD)) = 0$.

Proof. This is an application of [1, lemma IV.1.2]: if $\mathrm{h}^0(C, \mathcal{O}_C(D)) \geq 1$ for some divisor D we have $\deg(D) \geq 0$. To prove this statement, observe that we can obtain an *effective* divisor D' linearly equivalent to D, because we can use the non-zero global sections of $\mathcal{O}_C(D)$ and take the *divisor of zeroes* [1, proposition II.7.7]. Hence $\deg(D) = \deg(D') \geq 1$.
So if $n \deg(D) > \deg(\mathrm{K}_C)$ we get $\deg(\mathrm{K}_C - nD) \leq -1$ and therefore

$$\mathrm{h}^0(C, \mathcal{O}_C(\mathrm{K}_C - nD)) = 0. \tag{2.4}$$

□

So if $\mathrm{K}_C - nD$ becomes "sufficiently negative" its H^0 will vanish, hence we have solved the Riemann–Roch problem!

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[1] E.g., the disjoint union of two $\mathbb{P}^1_k$'s, because its arithmetic genus equals

$$1 - \chi(\mathbb{P}^1_k \sqcup \mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k \sqcup \mathbb{P}^1_k}) = 1 - (2 - 0) = -1. \tag{2.2}$$

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2.1.4 The degree of the canonical divisor

The canonical divisor still might be a mysterious beast. But Riemann–Roch at least tells us its degree: applying it to K_C and using the definition of the genus we get

$$g - 1 = \deg(\mathrm{K}_C) + 1 - g \tag{2.5}$$

i.e. $\deg(\mathrm{K}_C) = 2g - 2$.
So for $\mathbb{P}^1_k$, where the genus is 0, we get that the canonical divisor is the -2∞ we've seen before. For elliptic curves (i.e. curves of genus 1) we get that the canonical divisor has degree 0, and again because the genus is 1 we know that there are (only) the constants as differentials, hence the canonical divisor is equivalent to 0.

2.1.5 Rational function with pole in a point

The Riemann–Roch problem has several special cases. One of them is the following situation [1, exercise IV.1.1]:

> There exists a nonconstant rational function which is regular everywhere except at a given point p.

It suffices to take $D = p$ in the context of the Riemann–Roch problem, and as soon as $n > \deg(\mathrm{K}_C)$ we have a non-zero global section for $\mathcal{O}_C(np)$. By the previous results on the degree of the canonical divisor we can easily understand how high we should at least go:

1. if $C = \mathbb{P}^1_k$ then $n = 1$ suffices as the degree of the canonical divisor is -2, our point $p \in \mathbb{P}^1_k$ corresponds to an $a \in \mathbb{C}$ on some affine chart, the desired rational function with a pole in p is nothing but $(z - a)^{-1}$;
2. if C is an elliptic curve the degree of the canonical divisor is 0, so again $n = 1$ suffices;
3. for higher genus curves the choice of $n > 2g - 2$ will always give us a rational function, but in general a lower number could do. This is discussed further in the paragraph on Weierstrass gaps.

2.1.6 Rational functions with poles in several points

In a completely analogous manner we can find rational functions with poles in any number of points, as long as we don't put a bound on the multiplicity of the pole. This is [1, exercise IV.1.2]. At this rate we will have solved all the exercises of this section in a whim[2]

2.1.7 Weierstrass points

We had the Riemann–Roch problem for a divisor of the form np, where we were interested in the numbers $\mathrm{h}^0(C, \mathcal{O}_C(np))$. These numbers count the rational functions with prescribed behaviour. If $n > \deg(\mathrm{K}_C) = 2g - 2$ we had a complete knowledge

[2] I won't do this though, but you are cordially invited to do them as they are interesting and not as frightening as most exercises in Hartshorne's book.

n	-1	0	1	2	...	$2g-2$	$2g-1$	$2g$	...
$\mathrm{h}^0(C,\mathcal{O}_C(np)$	0	1	?	?	?	?	g	$g+1$	...

Table 2.1: Behaviour of $\mathrm{h}^0(C,\mathcal{O}_C(np)$

about the behaviour. On the other hand, if $n \leq -1$ we have no global sections, whereas $n=0$ yields the constants. For $n=2g-1$ on the other hand we get

(2.6) $\mathrm{h}^0(C,\mathcal{O}_C((2g-1)p)) = 2g-1-g+1 = g$

and for $n = 2g-1+k$ we get $g+k$ as the correction term will always be zero. Hence we get the following table We also know that

1. the numbers that we fill in have to be increasing: if a pole of order at most n in p exists, it surely exists if we allow poles of order $n+1$;
2. they can moreover only increase by at most 1: if f and g are rational functions with a pole of the same order, then there exists a constant c such that $f+cg$ has a pole of lower order (by cancelling the leading term in the local expression for f and g).

For low genera we also know what happens:

1. $g=0$ has no missing terms, it is (starting from $n=-1$)

   (2.7) $0,1,2,3,4,5,\ldots$

2. $g=1$ has no missing terms, it is (starting from $n=-1$)

   (2.8) $0,1,1,2,3,4,\ldots$

3. $g=2$ has one missing term, it is (starting form $n=-1$)

   (2.9) $0,1,1,?,2,3,\ldots$

What we don't know is whether the sequence of numbers will depend on the choice of p (it does). It is a bit mysterious at first sight, but on a genus 2 curve there will be exactly 6 points[3] for which the missing term is 2, whereas all the others have missing terms 1. This behaviour is the same for all higher genera: there are finitely many points in which the behaviour is "not as expected", where the expectation is that the sequence starts with $g+1$ copies of 1, after which it increases by 1 each time. Moreover, the number of exceptional points is $g(g^2-1)$ (if one weighs them).

Definition 26. The points with "exceptional behaviour" as explained before are called *Weierstrass points*

The sequence of missing terms associated to such a Weierstrass point is interesting to study, and depends on the type of curve we are considering. This ties in with the use of Riemann–Roch in the classification of curves. In the case of hyperelliptic curves (in arbitrary genus) the Weierstrass points are exactly the ramification points (with their correct weights!) but in general it's harder to say what they correspond to geometrically.

[3]These 6 points correspond to the ramification points of the degree 2 map to $\mathbb{P}^1_k$: every genus 2 curve is hyperelliptic, i.e. given by $y^2 = f(x)$ for f of degree 5 or 6.

2.1.8 The group law on elliptic curves

The Riemann–Roch theorem is also useful in proving that an elliptic curve E has a group structure. If one wishes to construct the group law by purely geometric notions proving the associativity is a bit hard (which is odd, because most of the times this axiom is rather easy to check). But there is an obvious group structure on the divisors, which we can put to good use.

To do so we will need $\mathrm{Pic}^0(E)$, this is the subgroup of degree-zero elements of $\mathrm{Pic}(E)$ which is the group of all divisors modulo linear equivalence. The divisor of a rational function has degree zero, hence we have defined

(2.10) $\mathrm{Pic}^0(E) = \{D \in \mathrm{Div}(E) \mid \deg(D) = 0\}/\{\div(f) \mid f \in k(E)\}$.

Then we pick a neutral element $p_0 \in E$. This yields the map

(2.11) $E \mapsto \mathrm{Pic}^0(E) : p \mapsto \mathcal{O}_E(p - p_0)$.

To see that this is a surjection, we take D a divisor of degree 0. We wish to show that there exists a unique point $p \in E$ such that D is linearly equivalent to $p - p_0$. As the degree of the canonical divisor is 0 we get $\deg(\mathrm{K}_E - D - p_0) = -1$, so $\mathrm{h}^0(E, \mathcal{O}_E(\mathrm{K}_E - D - p_0)) = 0$. Hence $\mathrm{h}^0(E, \mathcal{O}_E(D + p_0)) = 1$. As before we can find an effective divisor linearly equivalent do $D + p_0$, but as the the dimension is 1 and by applying [1, proposition II.7.7] we get that this divisor is unique (the linear system is zero-dimensional). As the degree of this divisor is 1 we get a single point p, i.e. the divisor p is rationally equivalent to $D + p_0$, or $D = p - p_0$ in $\mathrm{Pic}^0(E)$.

2.1.9 Classification of curves

Time and space don't permit me to write anything about it, but Riemann–Roch is crucial in tackling classification problems for curves.

2.2 Applications of Riemann–Roch for surfaces

2.2.1 Irregularity of a surface

Let S be a projective nonsingular surface. Its geometric genus is

(2.12) $\mathrm{p}_{\mathrm{geom}}(S) = \mathrm{h}^0(S, \omega_S) = \mathrm{h}^2(S, \mathcal{O}_S)$

whereas its arithmetic genus is $\mathrm{p}_{\mathrm{arith}}(S) = \mathrm{h}^2(S, \mathcal{O}_S) - \mathrm{h}^1(S, \mathcal{O}_S)$. Hence

(2.13) $\mathrm{p}_{\mathrm{geom}}(S) \geq \mathrm{p}_{\mathrm{arith}}(S)$.

Originally the Italian school thought (without knowing what sheaf cohomology was, they did all these things in more classical terms) that there was an equality. When they found out there are surfaces that have the strict inequality they dubbed the difference the *irregularity*.

2.2.2 Vanishing of H^2

As in the case of curves we have a criterion for the vanishing of a cohomology group if a certain numerical criterion is satisfied [1, lemma V.1.7].

Proposition 27. Let H be an ample divisor on a surface S. Then there exists an integer n_0 (in fact, it is $\mathrm{K}_S \cdot H$) such that, if D is a divisor for which $D \cdot H > n_0$, then $\mathrm{H}^2(S, \mathcal{L}(D)) = 0$.

Recall that the situation for curves asks $\deg D > n_0 = 2g_C - 2$ for $\mathrm{H}^1(C, \mathcal{L}(D))$ to be zero.

These vanishing results are pervasive throughout algebraic geometry. As we've seen in this talk the correct notion to study is the Euler–Poincaré characteristic, which incorporates all cohomological information. If one is interested in a single number though (which one often is) this requires these vanishing results in order to obtain a conclusion on this single number.

2.2.3 Invariants of special surfaces

Surfaces come in a wealth of families or shapes. Often we can find interesting (numerical) information for a specific choice of surface(s).

1. If S is a surface of degree d in $\mathbb{P}^3_k$ (i.e. defined by a homogeneous equation of degree 4 in four variables such that the Jacobian matrix is nonsingular) then the self-intersection of the canonical divisor K_S is given by

   (2.14) $\mathrm{K}_S \cdot \mathrm{K}_S = d(d-4)^2.$

   Hence this number depends on d, in the same way $\deg(\mathrm{K}_C)$ depended on g.

2. If S is again a surface of degree d in $\mathbb{P}^3_k$ such that it contains a straight line $C = \mathbb{P}^1_k$ then $C \cdot C = 2 - d$. Hence we get negative self-intersection if $d \geq 3$, which at first is a truly counterintuitive thing. In characteristic 0 one can find such a surface for any choice of d.

3. If $S = C \times C'$ is a product of two curves of genus g and g' respectively, then

   (2.15) $\mathrm{K}_{C \times C'} = 8(g-1)(g'-1).$

References

[1] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics 52. Springer, 1977, pp. xvi+496. ISBN: 978-0-387-90244-9.

Lecture 3

Derived categories and Grothendieck duality

Abstract

These are the notes for my third lecture on Grothendieck duality in the ANAGRAMS seminar. We (finally) come to a statement of Grothendieck duality. In order to do so we first review derived categories, from the viewpoint of someone who has already touched homological algebra in the usual sense [9]. After this quick reminder some motivation for considering a possible generalisation of Serre duality is discussed, after which the full statement of Grothendieck duality (in various incarnations) is given. To conclude some applications of Grothendieck duality are discussed, from my point of view on the subject. I hope these serve both as a motivation for Grothendieck duality and as a motivation to study these interesting subjects, regardless from their relationship with Grothendieck duality.

Contents

3.0 Reminder on derived categories

3.0.1 Derived functors

The main idea behind derived categories is to make working with derived functors more natural. Recall that given a left (or right) exact functor between abelian categories one can determine its derived functors, which form a family of functors. These functors measure the extent to which the original functor is not exact, and they can give interesting algebraic or geometric information (depending on the original choice of functor).

Example 28. So far we have used just one derived functor, which was sheaf cohomology. The left-exact functor under consideration is $\Gamma(X,-)$, its derived functors $\mathrm{H}^i(X,-)$.

There are other examples.

Example 29. Global sections are a special case of pushforward: if $f : X \to \operatorname{Spec} k$ is the structural morphism then $f_*(-) = \Gamma(X,-)$. We obtain a sheaf on $\operatorname{Spec} k$, which is nothing but a vectorspace, the only non-empty open set has $\Gamma(f^{-1}(\operatorname{Spec} k),-)$ as its sections. We can conclude that f_* will not be right-exact in general as it is a generalisation of global sections.

Example 30. Another well-known left exact functor is $\mathrm{Hom}_A(M,-)$, which is an endofunctor on the abelian category A-Mod for A a commutative ring and M an A-module), whose right-derived functors are the Ext-functors. These have a down-to-earth interpretation as extensions, by the Yoneda Ext-construction.

Example 31. Adjoint to $\mathrm{Hom}_A(M,-)$ we have $-\otimes_A M$, whose left-derived functors are the Tor-functors.

3.0.2 Derived categories

The goal is to capture all of these in one single *total derived functor*. So instead of working with the family $(\mathrm{R}^n F)_{n\in\mathbb{N}}$ one wants to construct a functor $\mathbf{R}F$ replacing the whole family.

To calculate (co)homology one uses injective (or projective, or flat, or flabby, or ... depending on the context) resolutions. So instead of using a single object, it is natural to consider a whole (co)chain complex of objects. That is why, instead of using an abelian category $\mathcal{A}$ (take for example $\mathcal{A} = \mathrm{Coh}/X$ the abelian category of coherent sheaves on a scheme), one uses $\mathrm{Ch}(\mathcal{A})$: the abelian category of (co)chain complexes over $\mathcal{A}$.

Because the calculation of (co)homology is invariant up to homotopy equivalence, we construct the category $\mathbf{K}(\mathcal{A})$ by identifying morphisms in $\mathrm{Ch}(\mathcal{A})$ which are homotopy equivalent. This is an intermediate step which can be skipped, but it helps in proving the main properties of the resulting object.

The final step in the construction is the most technical one, and consists of inverting the quasi-isomorphisms to obtain the *derived category*. Recall that a quasi-isomorphism is a morphism which induces isomorphisms in the (co)homology, i.e. if $f : A^\bullet \to B^\bullet$ is a morphism such that $\mathrm{H}^n(A^\bullet) \cong \mathrm{H}^n(B^\bullet)$ for all n then we would like $A^\bullet$ and $B^\bullet$ to be isomorphic in our desired derived category. This way, an object becomes isomorphic to its resolution. The way to obtain this is analogous to the localisation of a ring: we formally add inverses. That this construction works as intended follows from the Gabriel–Zisman theorem.

To summarise, the construction goes through the following steps

1. pick an abelian category $\mathcal{A}$ (Coh/X, Qcoh/X or just A-Mod if you like);
2. consider the abelian category of (co)chain complexes over $\mathcal{A}$;
3. construct the category of (co)chain complexes $\mathbf{K}(\mathcal{A})$ over $\mathcal{A}$ by identifying the homotopy equivalences in $\mathrm{Ch}(\mathcal{A})$;
4. construct the derived category $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ by inverting the quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$.

Instead of considering all (co)chain complexes, we can also consider complexes which satisfy a certain boundedness assumption. We could ask for (the cohomology of) the complexes to be

1. bounded on both sides (denoted $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$),
2. bounded below or above (denoted $\mathbf{D}^{+}(\mathcal{A})$ resp. $\mathbf{D}^{-}(\mathcal{A})$),
3. concentrated in positive or negative degrees (denoted $\mathbf{D}^{\geq 0}(\mathcal{A})$ resp. $\mathbf{D}^{\leq 0}(\mathcal{A})$).

One interesting property of the derived category is that the Hom-functor for $\mathcal{A}$ turns into a device that knows all Ext^i.

Example 32. We have a canonical inclusion of $\mathcal{A}$ into $\mathbf{D}(\mathcal{A})$, by considering an object as a cochain complex in degree 0. Now we can shift objects: the object $A[i]$ in $\mathbf{D}(\mathcal{A})$ is the cochain complex such that A lives in degree i. Then we obtain the formula

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A, B[i]) \cong \mathrm{Ext}^i_{\mathcal{A}}(A, B). \tag{3.1}$$

To see why this is true: the object $B[i]$ is isomorphic to a shift of an injective resolution in $\mathbf{D}(\mathcal{A})$, hence the Hom in $\mathbf{D}(\mathcal{A})$ is nothing but a way of computing the derived functors of Hom.

Sometimes we'll denote $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A^\bullet, B^\bullet)$ by $\mathbf{R}\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ to save a little on the notation.

3.1 Grothendieck duality

3.1.1 Motivation

There are several ways of motivating Grothendieck duality, and the desire to generalise Serre duality[1]. Of course, the restriction on the classical Serre duality are rather severe: we want a smooth (or mildly singular) projective variety over a field, and a vector bundle. Can we do similar things:

1. for more general schemes?

[1] If unfamiliar with Serre duality, one is either invited to read the notes to the first lecture, or glance at the summary of Serre duality later on.

2. over more general base schemes?
3. for more general sheaves?

The answer will be yes, otherwise we wouldn't be discussing Grothendieck duality.

Adjoint functors A more down-to-earth (or less categorical) motivation for the form that Grothendieck duality often takes is given in [6, chapter 6]. Recall from the first lecture the statement of Serre duality, preceded by the required definition of a dualising sheaf, as given in [4].

Definition 33. Let X/k be a proper n-dimensional variety. A *dualising sheaf* for X is a coherent sheaf ω_X° together with a *trace morphism* $\mathrm{tr}\colon \mathrm{H}^n(X,\omega_X^\circ)\to k$, such that for all $\mathcal{F}\in\mathrm{Coh}/X$ the natural pairing

(3.2) $\mathrm{Hom}(\mathcal{F},\omega_X^\circ)\times\mathrm{H}^n(X,\mathcal{F})\to\mathrm{H}^n(X,\omega_X^\circ)$

composed with tr gives an isomorphism

(3.3) $\mathrm{Hom}(\mathcal{F},\omega_X^\circ)\cong\mathrm{H}^n(X,\mathcal{F})^\vee.$

Then the statement, where X admits a dualising sheaf, reads:

Theorem 34 (Serre duality)**.** Let X/k be a projective n-dimensional variety. Let ω_X° be its dualising sheaf. Then for all $i\geq 0$ and $\mathcal{F}\in\mathrm{Coh}/X$ we have functorial maps

(3.4) $\theta^i\colon\ \mathrm{Ext}^i(\mathcal{F},\omega_X^\circ)\to\mathrm{H}^{n-i}(X,\mathcal{F})^\vee$

such that θ^0 corresponds to tr. Moreover, if X is Cohen–Macaulay[2] the θ^i are isomorphisms for all $i\geq 0$ and $\mathcal{F}\in\mathrm{Coh}/X$.

We also obtained a corollary, which explains the name "duality": it relates H^i to H^{n-i}, and does so by using a duality of vector spaces.

Corollary 35. Let X be projective Cohen–Macaulay of (equi-)dimension n over k. Let $\mathcal{F}$ be a locally free sheaf on X. Then we have isomorphisms

(3.5) $\mathrm{H}^i(X,\mathcal{F})\cong\mathrm{H}^{n-i}(X,\mathcal{F}^\vee\otimes\omega_X^\circ)^\vee.$

This is just one of the many ways of writing the isomorphism. Another would be

(3.6) $\mathrm{Hom}_k\left(\mathrm{H}^i(X,\mathcal{F}),k\right)\cong\mathrm{H}^{n-i}\left(X,\mathcal{H}om(\mathcal{F},\omega_X^\circ)\right).$

We are almost where we want to be. The last step to take is "go relative". Which of course in this case is not that spectacular. So let's look at the structural morphism $f: X\to\operatorname{Spec}k$. We are working for a vector bundle $\mathcal{F}$ on X, so we could also look at a vector bundle on $\operatorname{Spec}k$, which is nothing but a finite-dimensional vectorspace V. Generalising the previous equation, and applying the tensor-Hom adjunction we obtain

(3.7)
$$\mathrm{Hom}_k\left(\mathrm{H}^i(X,\mathcal{F}),V\right)\cong\mathrm{H}^{n-i}\left(X,\mathcal{H}om(\mathcal{F},V\otimes_k\omega_X^\circ)\right)\cong\mathrm{Ext}^{n-i}\left(\mathcal{F},V\otimes_k\omega_X^\circ\right).$$

With a little imagination this looks like an adjunction:

[2] A technical condition that says that "mild singularities" are allowed. It means that each local ring has Krull dimension equal to the depth (we always have that depth is bounded above by Krull dimension), where depth corresponds to the length of a maximal regular sequence for the local ring itself. One can just read non-singular, which is the case we will need in later applications.

1. the cohomology groups $\mathrm{H}^i(X,\mathcal{F})$ can be taken together[3] to $\mathbf{R}f_*(\mathcal{F})$;
2. by the properties of the derived category we can take Ext's together into a Hom in the derived category, see (3.1).

Hence we can write Serre duality as

$$(3.8)\quad \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\operatorname{Spec} k)}\left(\mathbf{R}f_*(\mathcal{F}),V\right)\cong \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}\left(\mathcal{F},f^!(V)\right)$$

where $\mathbf{D}^{\mathrm{b}}(\operatorname{Spec} k)$ is the bounded derived category of finite-dimensional k-vector spaces and $\mathbf{D}^{\mathrm{b}}(X)$ is the bounded derived category of coherent sheaves on X (assume X is smooth).
Hence Serre duality asserts the existence of a dual

$$(3.9)\quad f^!\colon \mathbf{D}^{\mathrm{b}}(\operatorname{Spec} k)\to \mathbf{D}^{\mathrm{b}}(X)$$

which in this case is explicitly given by $-\otimes_k \omega_X^\circ$. But in this statement we could easily replace $f:X\to\operatorname{Spec} k$ by a more general $f:X\to Y$, and the existence of a right adjoint $f^!$ would still make sense!
A word on the notation $f^!$:

1. It is often pronounced "f upper shriek", and it's named "exceptional inverse image";
2. It *only* lives on the level of derived categories, unlike f_* and f^*, which get derived into $\mathbf{R}f_*$ and $\mathbf{L}f^*$, so in line with [SGA4$_3$, §3.1, éxposé XVIII] which says (in a slightly different context, but still valid)

    > N.B. La notation $\mathbf{R}f^!$ est abusive en ce que $\mathbf{R}f^!$ n'est en général pas le dérivé d'un foncteur $f^!$.

    For this reason I will just denote it by $f^!$, as is already done for instance in [5].

Dualising complexes So we know that there is some virtue in looking at the relative context, and that we will obtain an adjoint pair encoding Grothendieck duality. Another thing we could do is look at the dualising sheaf. Then we can motivate Grothendieck duality by considering another rather trivial situation: $\operatorname{Spec}\mathbb{Z}$, as was done in [5, §V.1].
There are two ways of taking the dual of an abelian group[4]:

1. the *Pontryagin dual* of a finite abelian group, which is given by the functor $\mathrm{Hom}_{\mathrm{Ab}}(-,\mathbb{Q}/\mathbb{Z})$;
2. the dual of a finitely generated free group, which is given by the functor $\mathrm{Hom}_{\mathrm{Ab}}(-,\mathbb{Z})$.

[3] The functor f_* in this case is nothing but global sections, as $f_*(\mathcal{F})$ evaluated on $\operatorname{Spec} k$ is $\Gamma(f^{-1}(\operatorname{Spec} k),\mathcal{F})$, see example 29.

[4] Remark that there is a typo in [5, §V.1], dualising finitely generated free groups requires $\mathbb{Z}$, not $\mathbb{Q}$. This makes the exposition less miraculous.

Each applied twice to the correct situation gives a abelian group isomorphic to the one you started with. We can consider these two dualising functors at the same time[5], by considering the complex

$$\dots \to 0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \to \dots \tag{3.10}$$

in $\mathbf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Ab})$, the derived category of bounded complexes inside $\mathbf{D}^{+}(\mathrm{Ab})$ whose cohomology is finitely generated. This 2-term complex is an injective resolution of $\mathbb{Z}$, as both groups are divisible! So it is isomorphic to $\mathbb{Z}$ in $\mathbf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Ab})$, and to perform computations in the derived category we can interchange them freely.

This yields the following proposition [5, proposition:V.1-1].

Proposition 36. The functor

$$\mathrm{D}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{fg}} \to \mathbf{D}^{\mathrm{b}}_{\mathrm{fg}} : M^\bullet \to \mathbf{R}\mathrm{Hom}^\bullet(M^\bullet, \mathbb{Z}) \tag{3.11}$$

is a contravariant endofunctor, and there is a natural equivalence

$$\eta\colon \mathrm{id}_{\mathbf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Ab})} \Rightarrow \mathrm{D} \circ \mathrm{D}\,. \tag{3.12}$$

Hence on the “small” category $\mathbf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Ab})$ sitting inside the bigger $\mathbf{D}^{+}(\mathrm{Ab})$ this duality functor is truly a duality. The small category corresponds to the bounded derived category of coherent sheaves, as by the usual mantra “coherent = finitely generated.”

Proof of proposition 36. We have

$$\mathrm{H}^i(\mathrm{D}(M^\bullet)) = \mathrm{H}^i(\mathbf{R}\mathrm{Hom}^\bullet(M^\bullet, \mathbb{Z})) = \mathrm{Ext}^i(M^\bullet, \mathbb{Z}) \tag{3.13}$$

hence if $M^\bullet$ has finitely generated and bounded cohomology, so has $\mathrm{D}(M^\bullet)$. We obtain that D is a well-defined endofunctor.

The natural equivalence is defined in [5, lemma V.1.2] in an obvious way. To check that it is a natural equivalence: take a free resolution of $M^\bullet$. This means finding a surjection of a free abelian group onto $M^\bullet$ and repeating this process for the kernel of this map, and so on.

By [5, lemma I.7.1] it suffices to check it for $M^\bullet = \mathbb{Z}^r$ for some $r \geq 1$ as we only care about finitely generated cohomology. But things are additive, hence we can take $r = 1$. Now it suffices to observe that

$$\mathrm{Ext}^i(\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases} \tag{3.14}$$

which gives the desired natural equivalence. □

3.1.2 The ideal theorem

The first thing we can consider as a form of Grothendieck duality is [5, Ideal theorem on page 6]. This summarises what one tries to prove to be able to speak of a “Grothendieck duality result”. After the statement we will collect some contexts in which we can prove this.

[5] This is where the lack of a miracle occurs.

Theorem 37 (Ideal theorem).

1. For every morphism $f: X \to Y$ of finite type[6] of preschemes[7] there is a functor

   (3.15) $f^!: \mathbf{D}(Y) \to \mathbf{D}(X)$

   such that

   a) if $g: Y \to Z$ is a second morphism of finite type, then $(g \circ f)^! = f^! \circ g^!$;
   b) if f is a smooth morphism, then

      (3.16) $f^!(\mathcal{G}) = f^*(\mathcal{G}) \otimes \omega$,

      where $\omega = \Omega^n_{X/Y}$ is the sheaf of highest order differentials;
   c) if f is a finite[8] morphism, then

      (3.17) $f^!(\mathcal{G}) = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G})$.

2. For every proper[9] morphism $f: X \to Y$ of preschemes, there is a *trace morphism*

   (3.18) $\mathrm{Tr}_f : \mathbf{R}f_* \circ f^! \Rightarrow \mathrm{id}$

   of functors from $\mathbf{D}(Y)$ to $\mathbf{D}(Y)$ such that

   a) if $g: Y \to Z$ is a second proper morphism, then $\mathrm{Tr}_{g\circ f} = \mathrm{Tr}_g \circ \mathrm{Tr}_f$;
   b) if $X = \mathbb{P}^n_Y$, then Tr_f is the map deduced from the canonical isomorphism $\mathrm{R}^n f_*(\omega) \cong \mathcal{O}_Y$;
   c) if f is a finite morphism, then Tr_f is obtained from the natural map "evaluation at one"

      (3.19) $\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G}) \to \mathcal{G}$.

3. If $f: X \to Y$ is a proper morphism, then the *duality morphism*

   (3.20) $\Theta_f: \mathbf{R}\mathrm{Hom}_X(\mathcal{F}, f^!(\mathcal{G})) \to \mathbf{R}\mathrm{Hom}_Y(\mathbf{R}f_*\mathcal{F}, \mathcal{G})$

   obtained by composing the natural map[10] above with Tr_f, is an isomorphism for $\mathcal{F} \in \mathbf{D}(X)$ and $\mathcal{G} \in \mathbf{D}(Y)$.

[6]Recall that a morphism of finite type means that there exists an open affine covering of the codomain, such that the inverse images of these open sets admit a finite open affine covering, such that each of these rings is finitely generated over the open affine in the codomain.

[7]I have left this historic terminology in: what nowadays are called schemes were called preschemes in the early days of scheme theory. At first the philosophy was that we'd be mostly interested in separated schemes, which were called schemes, and not necessarily separated schemes were preschemes. In case you didn't know this little fact from the history of scheme theory, you know understand references to preschemes.

[8]Recall that a morphism is finite if there exists an open affine covering of the codomain such that the inverse image of each open affine is again affine, and moreover finite as a module over the original ring.

[9]Recall that a proper morphism between schemes is like a proper map of topological spaces, where inverse images of compact sets are again compact. As being compact in a non-Hausdorff context doesn't make much sense, algebraic geometers use a different definition of proper: the map $f: X \to Y$ between schemes is said to be *proper* if it is universally closed (i.e. for all $Y \to Z$ is $X \times_Y Z \to Z$ closed on the level of underlying topological spaces) and separated (i.e. $\Delta_f : X \to X \times_Y X$ is closed, which is an analogue of being Hausdorff).

[10]Obtained as the Yoneda pairing, see [5, page 5].

So the ideal theorem has 3 main themes:

1. the existence of the adjoint $f^!$;
2. the trace morphism for proper morphisms;
3. the duality for proper morphisms.

The first theme describes the functor $f^!$ in the cases where we know what it should be. The second theme describes what the counit adjunction should look like, while the third theme asserts that it actually is an adjunction. Remark that

1. We haven't specified $\mathbf{D}(X)$.
2. The role of $\mathbf{D}(X)$ will change depending on
   a) the type of schemes we are considering;
   b) the type of maps we are considering;
   c) the theme we are considering.

The question is: in which situations can we prove this ideal theorem? The answer (as per [5], nowadays it is more general, this will be discussed in the next lecture) is:

1. noetherian schemes of finite Krull dimensions and morphisms which factor through a suitable projective space [5, §III.8, §III.10, §III.11]: all statements are applied for $\mathbf{D}^+_{\mathrm{qc}}(-)$ except the last in which case $\mathcal{F}$ lives in $\mathbf{D}^-_{\mathrm{qc}}(X)$;
2. noetherian schemes which admit dualizing complexes (see [5, §V.10], it implies finite Krull dimension) and morphisms whose fibres are of bounded dimension [5, §VII.3]: all statements are applied to $\mathbf{D}^+_{\mathrm{coh}}(-)$ except the last in which case $\mathcal{F}$ lives in $\mathbf{D}^-_{\mathrm{qc}}(X)$;
3. noetherian schemes of finite Krull dimension and smooth morphisms [5, §VII.4]: the results in 1 are applied to $\mathbf{D}^+_{\mathrm{qc}}(-)$, in 2 to $\mathbf{D}^{\mathrm{b}}_{\mathrm{qc}}(-)$ and in 3 to $\mathcal{F}$ in $\mathbf{D}^-_{\mathrm{qc}}(X)$ and $\mathcal{G}$ in $\mathbf{D}^{\mathrm{b}}_{\mathrm{qc}}(Y)$;
4. noetherian schemes and arbitrary morphisms [5, appendix], but only statements 1a, 2a and 3: the results are applied to $\mathbf{D}(\mathrm{Qcoh}(-))$.

In tables 3.1 and 3.2 we have fitted this information in a nice overview.

Remark 38. The *duality morphism* (3.20) also has a relative version [2, p. 3.4.4], which reads

$$\underline{\Theta}_f : \mathbf{R}f_* \,\mathbf{R}\mathcal{H}om^\bullet_X(\mathcal{F}^\bullet, f^!(\mathcal{G}^\bullet)) \to \mathbf{R}\mathcal{H}om^\bullet_Y(\mathbf{R}f_*(\mathcal{F}^\bullet), G^\bullet). \tag{3.21}$$

Taking global sections yields the result mentioned in the ideal theorem.

3.1.3 What about dualising complexes?

The ideal theorem as stated here does not mention dualising objects. But in the case of Serre duality we really phrased things in terms of our "magic object" ω_X° which made the theory work. In the approach to Grothendieck duality of Hartshorne these dualising objects still play an important role (in the next lecture we will see approaches in which the role of this explicit object is greatly diminished), and trying to get a hold on them is the main difficulty. The problem in handling dualising complexes is that they are objects in the derived category, and there is the usual mantra that triangulated categories do not allow gluing, so a straightforward local approach doesn't work.

In the proof of Grothendieck duality one tries to keep track of what the dualising object looks like. Recall that in the case of Riemann–Roch this dualising object was Ω_C^1, and in the context of Serre duality we used $\Omega^n_{\mathbb{P}^n_k/k}$ for projective space, and a $\mathcal{E}xt$ construction for the general case of a projective variety.

In the situation of Grothendieck duality where we have used derived categories everywhere one can ask what ω_X° looks like. The answer is given in table 3.3, based on [5, §V.9].

We observe that the philosophy is "the nicer X, the nicer ω_X°".

3.2 Applications of Grothendieck duality

Due to time constraints, both in preparing these notes and actually lecturing about them, the following list of applications is not as worked out as I want it to be.

3.2.1 The yoga of six functors

Coherent duality The notion of Grothendieck duality that we have seen so far is in the following situation:

1. (quasi)coherent sheaves;
2. Zariski topology for schemes;

Étale cohomology But one can consider other contexts too. In the study of étale cohomology we have:

1. torsion sheaves;
2. étale topology for schemes.

Poincaré–Verdier duality In the case of manifolds and locally compact spaces we have Poincaré–Verdier duality:

1. sheaves of abelian groups;
2. locally compact spaces.

situation	X and Y	f
1	noetherian, finite Krull dimension	factor through $\mathbb{P}^n_Y$
2	noetherian with dualizing complex	
3	noetherian, finite Krull dimension	smooth
4	noetherian	

Table 3.1: Description of the 4 situations for Grothendieck duality in [5]

property	situation	$\mathbf{D}(X)$	$\mathbf{D}(Y)$
existence of $f^!$	1		$\mathbf{D}^+_{\mathrm{qc}}(Y)$
	2		$\mathbf{D}^+_{\mathrm{coh}}(Y)$
	3		$\mathbf{D}^{\mathrm{b}}_{\mathrm{qc}}(Y)$
	4		$\mathbf{D}^+(\mathrm{Qcoh}/Y)$
trace morphism	1		$\mathbf{D}^+_{\mathrm{qc}}(Y)$
	2		$\mathbf{D}^+_{\mathrm{coh}}(Y)$
	3		$\mathbf{D}^{\mathrm{b}}_{\mathrm{qc}}(Y)$
	4		$\mathbf{D}^+(\mathrm{Qcoh}/Y)$
duality	1	$\mathbf{D}^-_{\mathrm{qc}}(X)$	$\mathbf{D}^+_{\mathrm{qc}}(Y)$
	2	$\mathbf{D}^-_{\mathrm{qc}}(X)$	$\mathbf{D}^+_{\mathrm{coh}}(Y)$
	3	$\mathbf{D}^-_{\mathrm{qc}}(X)$	$\mathbf{D}^{\mathrm{b}}_{\mathrm{qc}}(Y)$
	4	$\mathbf{D}(\mathrm{Qcoh}/X)$	$\mathbf{D}^+(\mathrm{Qcoh}/Y)$

Table 3.2: Overview of the configuration of the derived categories for each of the three parts of a Grothendieck duality context

how nice is X?	how nice is ω_X°?
X smooth	$\omega_X^\circ = \bigwedge^{\dim X} \Omega_X[\dim X]$
X Gorenstein	ω_X° shift of a line bundle by $\dim X$
X Cohen–Macaulay	ω_X° shift of a sheaf by $\dim X$
X arbitrary	ω_X° is a complex

Table 3.3: Comparison of singularness of X and the look of ω_X°

Formalism In formalising the properties that are similar in each of these contexts we see that

- we are considering *image functors* of sheaves;
- we are using the *closed monoidal structure* of the category of sheaves.

In the general situation of a “six functors formalism” we can identify the following functors [1], for $f : X \to Y$ a morphism in some category, and $\mathcal{C}(X)$ some category of sheaves associated to X.

notation	name	signature
f^*	inverse image	$f^*\colon \mathcal{C}(Y) \to \mathcal{C}(X)$
f_*	direct image	$f_*\colon \mathcal{C}(X) \to \mathcal{C}(Y)$
$f_!$	exceptional direct image	$f_!\colon \mathcal{C}(X) \to \mathcal{C}(Y)$
$f^!$	exceptional inverse image	$f^!\colon \mathcal{C}(Y) \to \mathcal{C}(X)$
$\mathcal{H}om(-,-)$	internal Hom	$\mathcal{C}(X) \times \mathcal{C}(X) \to \mathcal{C}(X)$
$- \otimes -$	internal tensor product	$\mathcal{C}(X) \times \mathcal{C}(X) \to \mathcal{C}(X)$

Depending on the context the adjective exceptional is sometimes replaced by the adjectives proper or twisted. The existence of some of the functors is not required to be universal in f, e.g. the exceptional direct and inverse image are only required to exist for separated maps of finite type between schemes.

We have relationships between these functors. These are (being a bit sloppy and not mentioning all of them):

1. the adjunctions: $f^* \dashv f_*$, $f_! \dashv f^!$ and $- \otimes C \dashv \mathcal{H}om(C,-)$;
2. the existence of a natural transformation $\alpha_f : f_! \Rightarrow f_*$;
3. base change isomorphisms intertwining inverse and exceptional direct image, and direct and exceptional direct image;

In the situation of coherent duality we moreover have:

1. the adjunction (with some abuse of notation, dropping $\mathbf{R}$)

   (3.22) $f_* \dashv f^!$

   if $f : X \to Y$ is a proper map between the correct type of schemes[11];
2. compatibilities between dualising functors and the image functors (i.e. compatibilities between the closed monoidal structure and the relative struture).

Hence we have several possibilities to go continue our study of Grothendieck duality:

1. develop Grothendieck duality and the six functors as a formal property of monoidal categories (won’t be done here);
2. develop these five or six functors into an interesting calculus.

This second option is exactly what we’re going to do in the next section. The compatibilities not mentioned explicitly will return there.

[11] Rather, we have $f_* = f_!$ in this situation. But in general these are different, so we really have six functors and not just five.

3.2.2 Fourier–Mukai transforms

Using this formalism of five (or six) functors we can get an interesting "calculus of derived functors". Some of these properties have been stated in the previous subsection, but we will now repeat them. The goal is to show that on the level of derived categories one gets lots of compatibilities which can be useful for computations, especially after we have discussed Orlov's existence result.

We will consider $f : X \to Y$ a morphism of projective schemes over a field k which is the context of [7]. The following are formal, in the sense that they are generalisations of the underived formulas.

projection formula

f proper, $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X)$ and $\mathcal{G}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/Y)$

$$\mathbf{R}f_*(\mathcal{F}^\bullet) \otimes^{\mathbf{L}} \mathcal{G}^\bullet \xrightarrow{\sim} \mathbf{R}f_*\left(\mathcal{F}^\bullet \otimes^{\mathbf{L}} \mathbf{L}f^*(\mathcal{E}^\bullet)\right) \tag{3.23}$$

$\mathbf{L}f^*$ and $\otimes^{\mathbf{L}}$ commute

$\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/Y)$

$$\mathbf{L}f^*(\mathcal{F}^\bullet) \otimes^{\mathbf{L}} \mathbf{L}f^*(\mathcal{G}^\bullet) \xrightarrow{\sim} \mathbf{L}f^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}} \mathcal{G}^\bullet); \tag{3.24}$$

$\mathbf{L}f^*$ and $\mathbf{R}f_*$ adjunction

f projective, $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X)$, $\mathcal{G}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/Y)$

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X)}\left(\mathbf{L}f^*(\mathcal{G}^\bullet), \mathcal{F}^\bullet\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X)}\left(\mathcal{G}^\bullet, \mathbf{R}f_*(\mathcal{F}^\bullet)\right); \tag{3.25}$$

$\otimes^{\mathbf{L}}$ and $\mathbf{R}\mathcal{H}om$ adjunction

X smooth and projective, $\mathcal{F}^\bullet, \mathcal{G}^\bullet, \mathcal{H}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X)$

$$\begin{aligned} \mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \otimes^{\mathbf{L}} \mathcal{G}^\bullet &\cong \mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet \otimes^{\mathbf{L}} \mathcal{H}^\bullet), \\ \mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathbf{R}\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{H}^\bullet)) &\cong \mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet \otimes^{\mathbf{L}} \mathcal{G}^\bullet, \mathcal{H}^\bullet), \\ \mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet \otimes^{\mathbf{L}} \mathcal{H}^\bullet) &\cong \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{F}^\bullet), \mathcal{H}^\bullet); \end{aligned} \tag{3.26}$$

global sections and $\mathbf{R}\mathcal{H}om$

$\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X)$

$$\mathbf{R}\Gamma \circ \mathbf{R}\mathcal{H}om_X(\mathcal{F}^\bullet, -) = \mathbf{R}\mathrm{Hom}(\mathcal{F}^\bullet, -); \tag{3.27}$$

$\mathbf{L}f^*$ and $\mathbf{R}\mathcal{H}om$ commute

$\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathbf{D}^{\mathrm{b}}(Y)$

$$\mathbf{L}f^*\left(\mathbf{R}\mathcal{H}om_Y(\mathcal{F}^\bullet, \mathcal{G}^\bullet)\right) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_X\left(\mathbf{L}f^*(\mathcal{F}^\bullet), \mathbf{L}f^*(\mathcal{G}^\bullet)\right); \tag{3.28}$$

flat base change

for

$$
\begin{tikzcd}
Y' \arrow[r, "v"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\
X' \arrow[r, "u"'] & X
\end{tikzcd} \tag{3.29}
$$

with f proper and u flat, $\mathcal{F}^\bullet \in \mathbf{D}(\mathrm{Qcoh}/Y)$

$$u^* \circ \mathbf{R}f_*(\mathcal{F}^\bullet) \xrightarrow{\sim} \mathbf{R}g_* \circ v^*(\mathcal{F}^\bullet). \tag{3.30}$$

But as we saw in the description of the general six functor formalism we have another functor at our disposal: $f^!$.
Or rather, in the context we are working in right now (smooth projective varieties over a scheme) we work with the dualising sheaf characterising the $f^!$, which makes things more explicit. So if $f : X \to Y$ is a morphism between such schemes we define explicitly

$$\omega_f \coloneqq \omega_X \otimes f^*(\omega_Y^\vee) \tag{3.31}$$

and

$$\dim(f) \coloneqq \dim X - \dim Y. \tag{3.32}$$

This means that $f^!$ is given by

$$f^! : \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/Y) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X) : \mathcal{F}^\bullet \mapsto \mathbf{L}f^*(\mathcal{E}^\bullet) \otimes^{\mathbf{L}} \omega_f[\dim(f)]. \tag{3.33}$$

Then we get some new compatibilities between our functors on derived categories, which are just a remanifestation of Grothendieck duality.

Grothendieck duality $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(X)$, $\mathcal{G}^\bullet \in \mathbf{D}^{\mathrm{b}}(Y)$

$$\mathbf{R}f_*\left(\mathbf{R}\mathcal{H}om\left(\mathcal{F}^\bullet, \mathbf{L}f^*(\mathcal{G}^\bullet) \otimes^{\mathbf{L}} \omega_f[\dim(f)]\right)\right) \cong \mathbf{R}\mathcal{H}om\left(\mathbf{R}f_*(\mathcal{F}^\bullet), \mathcal{G}^\bullet\right); \tag{3.34}$$

$\mathbf{R}f_*$ and $f^!$ adjunction $\mathcal{F}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X)$, $\mathcal{G}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/Y)$

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/Y)}\left(\mathbf{R}f_*(\mathcal{F}^\bullet), \mathcal{G}^\bullet\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X)}\left(\mathcal{F}^\bullet, f^!(\mathcal{G}^\bullet)\right). \tag{3.35}$$

The crux of all this is the following representability result. Recall that a *Fourier–Mukai functor* with kernel $\mathcal{P}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X \times Y)$ is given by

$$\Phi_{\mathcal{P}^\bullet} : \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y) : \mathcal{F}^\bullet \mapsto \mathbf{R}p_*\left(\mathbf{L}q^*(\mathcal{F}^\bullet) \otimes^{\mathbf{L}} \mathcal{P}^\bullet\right) \tag{3.36}$$

where $\mathbf{L}q^* = q^*$ as p, q are the projections on Y and X respectively. Hence this is a specific type of functor, given by geometric information. Then we have the following result, which says that lots of interesting functors are actually Fourier–Mukai transforms!

Theorem 39. Let X and Y be smooth projective varieties. Let

(3.37) $F\colon \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X) \to \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/Y)$

be a fully faithful exact functor. Then there exists a $\mathcal{P}^\bullet \in \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X \times Y)$ such that $F \cong \Phi_{\mathcal{P}^\bullet}$.

Because we can get strong results on Fourier–Mukai transforms (regardless of whether they are actually representing a functor as in the theorem or not) we have obtained an interesting "calculus of derived functors". This is an important area of current research, from many different perspectives.

Some of the results in [7] which appeal immediately to Grothendieck duality are:

1. an explicit formula for the left and right adjoint [7, proposition 5.9];
2. braid group actions for spherical objects [7, lemma 8.21];
3. the study of flips and flops [7, §11.1];
4. semi-orthogonal decompositions of derived categories [7, §11.2];
5. ...

3.2.3 The moduli of curves

The paper that introduced *stacks* to the world [3] also applies Grothendieck duality right from the start. The goal is to study the *moduli space* $\mathcal{M}_g$ *of curves of genus* g, and show that it is irreducible, regardless of the choice of base field.

As they say themselves, the "key definition of the whole paper" is:

Definition 40. Let S be any scheme. Let $g \geq 2$. A *stable curve of genus* g over S is a proper flat morphism $\pi\colon C \to S$ whose geometric fibres are reduced, connected, 1-dimensional schemes C_s such that

1. C_s has only ordinary double points;
2. if E is a non-singular rational component of C_s, then E meets the other components of C_s in more than 2 points;
3. $\dim \mathrm{H}^1(\mathcal{O}_{C_s}) = g$.

So two aspects of Grothendieck duality come to mind: the relative situation, and the (mild) singularities. We get a canonical invertible sheaf $\omega_{C/S}$ on C, where C will act as a *family* of sufficiently nice curves to connect any two points in the moduli space, thus proving irreducibility.

One then proves the following properties of the dualising sheaf:

1. $\omega_{C/S}^{\otimes n}$ is relatively very ample for $n \geq 3$;
2. $\pi_*(\omega_{C/S}^{\otimes n})$ is locally free of rank $(2n-1)(g-1)$.

The proof of these properties uses the fact that we "almost" get a smooth curve of genus g, and we study the different irreducible components, together with the explicit manifestation of Grothendieck duality for curves with at most ordinary double points.

Hence we can conclude that, taking $n = 3$, we can realise a stable curve $C \to S$ as a family of curves inside $\mathbb{P}^{5g-6}$ such that the Hilbert polynomial of each point is $(6n-1)(g-1)$.

This yields the construction of a subscheme $\mathrm{H}_g \subseteq \mathrm{Hilb}^{(6n-1)(g-1)}_{\mathbb{P}^{5g-6}}$ of *tricanonically embedded stable curves*, i.e. the functor described by

$$\mathrm{Hom}_{\mathrm{Sch}}(S, \mathrm{H}_g) \cong \left\{ \left(\pi\colon C \to S \text{ stable; } \mathrm{Proj}\left(\pi_*(\omega_{C/S}^{\otimes 3}) \right) \cong \mathbb{P}_S^{5g-6} \right) \right\} / \cong \tag{3.38}$$

for a scheme S. By taking the quotient of the (open locus of smooth curves of the) scheme H_g by the PGL_{5g-6}-action we obtain a model for the moduli space of (smooth) curves, and hence we can try to compute things.

From this point on the proof does not use Grothendieck duality anymore, so I will end the summary here.

3.2.4 Other applications

Each of the following applications more than deserves a proper treatment. Unfortunately this is not possible here, due to lack of time, space and familiarity with the subject. They are here to show how diverse applications of Grothendieck duality can get. Any error in this list is due to my limited knowledge on the subject.

Local duality The study of local rings and singularities leads to working with Cohen–Macaulay rings and modules, and understanding these in as concrete terms as possible. It is related to representation theory as well.

Singularity categories This is another approach to studying singularities, now in a more global setting. It is similar to the previous application in some respects, but more alike studying Fourier–Mukai transforms and derived categories in others.

Noncommutative algebra The notion of dualising complex has a counterpart for noncommutative rings.

Noncommutative algebraic geometry The notion of Serre and Grothendieck duality leads to studying abstract Serre functors in triangulated or dg categories. This is also related to Calabi–Yau categories.

Arithmetic geometry The relative formalism also applies to arithmetic geometry, for example in studying Eisenstein ideals [8]. I know absolutely nothing about it.

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Lecture 4

Sketches of some of the proofs

Abstract

These are the notes for my fourth and final lecture on Grothendieck duality in the ANAGRAMS seminar. They are dedicated to an overview of some of the proofs in the literature. They are significantly more detailed than the exposé in the seminar. I describe:

1. Hartshorne's geometric proof,
2. Deligne's pro-objects categorical proof,
3. Neeman's proof based on Brown's representability,
4. Murfet's proof based on the mock homotopy category of projectives;

while the approaches by Lipman and Yekutieli–Zhang are merely mentioned. The goal of discussing these approaches is to highlight the two main contrasting aspects of Grothendieck duality:

1. local versus global;
2. categorical versus geometric.

The interplay of these makes the study of Grothendieck duality so interesting.

Contents

4.1 Hartshorne's proof: dualising and residual complexes

4.1.1 Introduction

The first proof of Grothendieck duality was given by Robin Hartshorne in 1966 [7], based on notes provided by Alexander Grothendieck in 1963. As the statement and proof require the use of derived categories, Jean–Louis Verdier's ongoing (at the time) work was included in the first two chapters of the book, and it was (as far as I can tell) the first published treatise of derived categories.

Issues with the proof This approach is the most geometric of them all, but also the most complicated. To quote Amnon Neeman [16]:

> [...] Since derived categories are basically unsuited for local computations, the argument turns out to be quite unpleasant.

If one reads the proof as outlined in [7] this will become clear: after introducing the required notions of derived categories in algebraic geometry (the first 100 pages) the proof takes 250 pages. These 250 pages also only summarise many important results on local cohomology and depend heavily on technical results in the EGA's.
Moreover, the proof from [7] is incomplete, and contains errors. Regarding the incompleteness the author himself says in [7, §II.5]:

> Now these examples are only three of many more compatibilities which will come immediately to the reader's mind. I could make a big list, and in principle could prove each one on the list. [...] And since the chore of inventing these diagrams and checking their commutativity is almost mechanical, the reader would not want to read them, nor I write them. [...]

Hence the reader is left with checking lots and lots of commutative diagrams, some of them depending on very subtle sign conventions in homological algebra!
Besides these intentional omissions there are some mistakes in the proofs. Nothing that can't be fixed though.

Trace maps and base change There is an important omission from the proof: the compatibility of the trace map for smooth morphisms with arbitrary base change. If $f : X \to Y$ is a proper, surjective, smooth map of schemes whose fibers are equidimensional of dimension n, then we had the *trace map* [7, §VII.4]

$$\gamma_f : \mathrm{R}^n f_*(\omega_{X/Y}) \to \mathcal{O}_Y \tag{4.1}$$

which is an isomorphism if f has geometrically connected fibers. Now let

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y \end{tikzcd} \tag{4.2}$$

be a cartesian diagram, then we get an isomorphism

$$u^*\left(\mathrm{R}^n f_*(\omega_{X/Y})\right) \to \mathrm{R}^n g_*\left(v^*(\omega_{X/Y})\right) \cong \mathrm{R}^n g_*(\omega_{X'/Y'}). \tag{4.3}$$

The desired compatibility then asserts that

$$\begin{tikzcd}
u^*\left(\mathrm{R}^n f_*(\omega_{X/Y})\right) \arrow[rr, "\cong"] \arrow[dr, "u^*(\gamma_f)"'] & & \mathrm{R}^n g_*(\omega_{X'/Y'}) \arrow[dl, "\gamma_g"] \\
& u^*(\mathcal{O}_Y) = \mathcal{O}_{Y'} &
\end{tikzcd} \tag{4.4}$$

is a commutative diagram. A nice discussion of the state of this base change compatibility can be found in [3, §1.1]. To summarise: the proof is left to the reader in [7, §VII.4], and its proof is highly non-trivial, which brings us by [3].

Companion to the proof The book [3] is written as a complement to the original proof, providing information on the omissions and fixing the numerous mistakes in the original proof. As the theory of derived categories was in its infancy, many things were still unclear, and this caused errors. Some of these are trivial to fix, others are severe and require a completely different proof. And to top things of: some (minor) mistakes have been found in Brian Conrad's book, but these don't require difficult fixes and a detailed erratum is available.

In total, one understands that if the proof and a companion to the proof take about 500 pages, it's hardly an easy proof.

4.1.2 Outline of the proof

Summarising 500 pages of proof is a rather non-trivial task to do, but I will try to outline my view on the proof and its structure. Hopefully this helps in tackling the proof, and identifying which parts could be of interest to the reader. I will do this by summarising the different chapters, later on some interesting points will be highlighted in separate paragraphs.

As mentioned before (but maybe not explicitly enough): this (geometric) approach to Grothendieck duality can be summarised by the following slogan.

> We define $f^!$ by looking for a dualising complex and *defining* the functor in terms of this complex.

The whole setup of the book should be considered in this point of view.

chapters 1 and 2 As derived categories were still in their infancy and there was not a published text available about them, they are first introduced and then applied to the situation of schemes. For a more up-to-date introduction one can look at [8, chapters 1–3].

chapter 3 The proof of Grothendieck duality for projective morphisms. In this "easy" case we can do more explicit computations, and control the dualising object. The idea is to factor sufficiently nice morphisms (see section 4.1.3) into

1. smooth morphisms,
2. finite morphisms;

and introduce the functor $f^!$ for each of these. The functor $f^!$ for a finite morphism is denoted $f^\flat$, the one for a smooth morphism is $f^\sharp$.

By checking compatibility of these two definitions (which are suggested by the Ideal theorem, as given in the previous lecture) one obtains a theory of $f^!$ for these nice morphisms [7, theorem III.8.7] (but not all the required properties for $f^!$). This is then used to obtain Grothendieck duality for *projective* morphisms [7, §III.9, III.10].

A similar idea of factoring morphisms into tractable ones occurs in section 4.2.

chapter 4 As discussed in the section on applications of Grothendieck duality there is an interesting notion of local duality, related to local cohomology. Before tackling global Grothendieck duality one has to understand what happens in the local case, as this is what is used to characterise the objects defined in the next chapters.

Historically, local duality is treated in [SGA2] and [6]. Remark that this second book are the notes (by Hartshorne) of a seminar (by Grothendieck) on local duality, and are originally from 1961, the same period as SGA2 (which was done in 1961–1962).

chapter 5 Recall that this approach to Grothendieck duality can be summarised by the following slogan.

> We define $f^!$ by looking for a dualising complex and *define* the functor in terms of this complex.

But the statement of Grothendieck duality doesn't mention dualising complexes explicitly. Hence it is not required to develop this machinery if one is looking for $f^!$, but it does give an explicit flavour to it. Moreover, this is the approach taken in the proof, so we do require some understanding of it.

In this chapter the machinery and properties of dualising complexes are discussed. The goal is to understand how dualising complexes relate to local duality, how this behaves with respect to singularities and how we can interpret the dualising complexes. Some of these properties are discussed in section 4.1.4.

chapter 6 Unfortunately, dualising complexes live in a derived category, and this is a non-local object [7, page 193]. To solve this problem *residual complexes* are introduced. These are a manifestation of dualising complexes in the non-derived category of chain complexes. A nice motivation for having a theory for both dualising and residual complexes is given on [3, pages 106–107]. In the case of a curve C these two manifestations are

1. the dualising complex $\Omega^1_{C/k}[1]$ (familiar from Riemann–Roch!);
2. the residual complex

$$(4.5)\quad \ldots \to i_{\xi,*}\left(\Omega^1_{X/k,\xi}\right) \to \bigoplus_{x\in X^0} i_{x,*}\left(\Omega^1_{X/k,\xi}/\Omega^1_{X/k,x}\right) \to 0 \to \ldots$$

where ξ is the generic point of C.

These complexes are quasi-isomorphic to eachother, but the residual complex is a bunch of injective hulls taken together, which can be taken to live in a non-derived category and still allow for computations. For more information on the definition see section 4.1.5.

Now a theory for embeddable morphisms and residual complexes is developed, using functors f^{y} and f^{z} for finite and smooth morphisms[1]. Their definitions depend on (pointwise) dualising complexes, but they are truly functors on the non-derived level. This chapter is the technical part of the book, which lots of compatibility checks.

chapter 7 In this chapter Grothendieck duality in its general form is finally proved. We have obtained many preliminary results, and this allows us to summarise the final proof [7, theorem VII.3.3] as follows:

1. Grothendieck duality for $\mathbf{R}\mathcal{H}om$ is local (and it implies the other statements), hence we reduce Y to the spectrum of a local ring (so the base is affine).
2. By some machinery of derived categories we can replace the complex by a single quasicoherent sheaf on X.
3. We can replace the quasicoherent sheaf on X by a coherent one using a direct limit argument.
4. As Y is affine the local statement for $\mathbf{R}\mathcal{H}om$ becomes a global statement for $\mathbf{R}\mathrm{Hom}$.
5. We check compatibility of the global statement with composition of two morphisms (this is not a part of the conceptual flow of the proof in my opinion). This is where residual complexes are required. One of the compatibilities requires a coherence condition, which explains the reduction in the third step.
6. Using noetherian induction on X we can assume that the theorem is proven for every $g\colon Z \to Y$ where $i\colon Z \to X$ is a closed immersion with $Z \neq X$ and $g = f \circ i$.
7. As X is proper over Y we apply Chow's lemma to find an X' which is projective over Y and a morphism $g\colon X \to X'$ which is an isomorphism over some non-empty open subset U. By the noetherian induction we can assume the theorem proven for the complement, which allows us to reduce the statement to the projective morphism in the factorisation of Chow's lemma.
8. Now we can apply the results we had for projective morphisms, and conclude.

The remainder of the chapter is dedicated to spelling out the trace map and making the duality result more explicit for proper smooth morphisms.

4.1.3 Embeddable morphisms

To proof Grothendieck duality one first proves it for embeddable morphisms.

[1] You are right, this is really weird notation.

Definition 41. Let S be a base scheme. A morphism $f: X \to Y$ is *embeddable* (over S) if there exists a smooth scheme P (over S) and a finite morphism

(4.6) $i: X \to P_Y = P \times_S Y$

such that $f = p_2 \circ i$.

Exactly how useful is this definition?

1. If $f: X \to Y$ is finite then f can be factored through $P = S$.
2. If $f: X \to Y$ is projective, where Y is quasicompact and admits an ample sheaf, then f can be factored through some $P = \mathbb{P}^n_Y$ [EGA II, II.5.5.4(ii)].

The main issue is that morphisms of finite type (a very general class of morphisms) are locally embeddable, but not globally so. Hence this approach does not yield a theory of $f^!$ in general. To overcome this issue we need the notion of dualising and especially residual complexes.

4.1.4 Dualising complexes

Recall from the description of duality on $\operatorname{Spec}\mathbb{Z}$ (see the previous lecture) that we had a "bounded complex of quasicoherent sheaves with coherent cohomology". In this case all the highbrow terminology boils down to "a bounded complex of abelian groups with finitely generated cohomology". As the complex we considered was an injective resolution of $\mathbb{Z}$ this condition is clearly satisfied.

By considering this particular complex we obtained a duality functor

(4.7) $\mathrm{D}: M^\bullet \to \mathbf{R}\mathrm{Hom}^\bullet(M^\bullet, \mathbb{Z})$

for $M^\bullet$ an object in $\mathbf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Ab}) = \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\operatorname{Spec}\mathbb{Z})$, i.e. applying the dual twice yields something functorially isomorphic to what you started with. This is completely analogous to the case of vectorspaces: one has to start with a finite-dimensional one to get the double dual isomorphic to the original one.

Generalising this we get to the following definition.

Definition 42. Let X be locally noetherian. A *dualising complex* is a complex $\mathcal{R}^\bullet \in \mathbf{D}^+_{\mathrm{coh}}(X)_{\mathrm{fid}}$ such that for each $\mathcal{F}^\bullet \in \mathbf{D}(X)$ the morphism

(4.8) $\eta: \mathcal{F}^\bullet \to \underline{\mathrm{D}} \circ \underline{\mathrm{D}}(\mathcal{F}^\bullet) = \mathbf{R}\mathcal{H}om^\bullet(\mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{R}^\bullet), \mathcal{R}^\bullet)$

is an isomorphism.

As is fashionable in algebraic geometry, we have turned our problem into a definition. But this is a remarkably interesting definition, as we can observe the following *properties*:

1. the complex $\mathcal{R}^\bullet$ is quasi-isomorphic to a bounded complex of quasicoherent injective sheaves, hence we get that $\underline{\mathrm{D}}: \mathbf{D}(X) \to \mathbf{D}(X)$ sends $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ to itself, and it interchanges $\mathbf{D}^+_{\mathrm{coh}}(X)$ and $\mathbf{D}^-_{\mathrm{coh}}(X)$;
2. if X is regular of finite Krull dimension then $\mathcal{O}_X$ is already a dualising complex [7, example V.2.2];

3. one can check whether $\mathcal{R}^\bullet$ is dualising at all the stalks of closed points of X [7, corollary V.2.3];
4. dualising complexes are preserved by $f^\flat$ and $f^\sharp$ [7, proposition V.2.4 and theorem V.8.3], hence for embeddable f we get that $f^!$ preserves dualising complexes, and this means that we can compute $f^!$ by the following isomorphism

   $$f^!(\mathcal{F}^\bullet) \cong \underline{\mathrm{D}}_X \circ \mathbf{L}f^* \circ \underline{\mathrm{D}}_Y(\mathcal{F}^\bullet) \tag{4.9}$$

   with

   $$\begin{aligned} &\underline{\mathrm{D}}_X(-) := \mathbf{R}\mathcal{H}om_X^\bullet(-, f^!(\mathcal{R}^\bullet)) \\ &\underline{\mathrm{D}}_Y(-) := \mathbf{R}\mathcal{H}om_Y^\bullet(-, \mathcal{R}^\bullet) \end{aligned} \tag{4.10}$$

   if $\mathcal{R}^\bullet$ is a dualising complex on Y and $f : X \to Y$ is embeddable [7, proposition V.8.5];
5. if more generally f is of finite type with Y noetherian and $\mathcal{R}^\bullet$ a dualising complex on Y then $f^!(\mathcal{R}^\bullet)$ will be one on X;
6. dualising complexes are unique up to tensoring with invertible sheaves and shifts [7, theorem V.3.1].

Regarding the question of its *existence*, one has the following necessary and sufficient conditions [7, §V.10]:

Sufficient conditions Hence, under which conditions can we prove the existence of a dualising complex?

1. X Gorenstein and of finite Krull dimension.
2. X of finite type over Y with Y admitting a dualising complex (which has X of finite type over a field k as a special case, hence we get Serre duality for arbitrary singularities).

Necessary conditions Hence, what properties of X does the existence of a dualising complex imply?

1. X has finite Krull dimension.
2. X is catenary.

As the proof of Grothendieck duality by constructing an explicit formula for $f^!$ depends on dualising complexes, one could hope for a more general result by taking a different approach. This will be discussed later on. First we have to discuss how one can use local duality and dualising complexes to obtain a global theory.

4.1.5 Residual complexes

The problem is that we cannot glue dualising complexes together, as the derived category is not a local object [7, page 193]. Hence we need to do some work to use our dualising complexes from the previous paragraph and obtain $f^!$.

The idea is to take a dualising complex $\mathcal{R}^\bullet \in \mathbf{D}^+_{\mathrm{coh}}(X)$ and turn it into an actual complex (i.e. in some $\mathrm{Ch}(X)$) which will be called the *residual complex*.

As we can glue actual complexes together, we can obtain a $f^!$ by gluing residual complexes together. Of course, we need to know that it doesn't matter whether we use dualising complexes or residual complexes. I.e. we will look for a functor $\mathrm{E}\colon \mathbf{D}^+_{\mathrm{coh}}(X) \to \mathrm{Ch}^+_{\mathrm{coh}}(\mathrm{Qcoh}_{\mathrm{inj}}/X)$ such that $\mathrm{Q}\circ\mathrm{E}(\mathcal{R}^\bullet) \cong \mathcal{R}^\bullet$, where Q is the quotient functor in the construction of the derived category, and $\mathrm{Ch}^+_{\mathrm{coh}}(\mathrm{Qcoh}_{\mathrm{inj}}/X)$ is a model for $\mathbf{D}^+_{\mathrm{coh}}(X)$.

The definition of a residual complex seems odd at first sight.

Definition 43. Let X be a locally noetherian prescheme. A *residual complex* $\mathcal{K}^\bullet$ on X is a bounded below complex of quasicoherent injective $\mathcal{O}_X$-modules with coherent cohomology, together with an isomorphism

$$\bigoplus_{p\in\mathbb{Z}} \mathcal{K}^p \cong \bigoplus_{x\in X} \mathrm{J}(x) \tag{4.11}$$

where $\mathrm{J}(x)$ is the quasicoherent injective $\mathcal{O}_X$-module given by the constant sheaf with values in an injective hull of $k(x)$ over $\mathcal{O}_{X,x}$ on $\mathrm{cl}(\{x\})$, and zero elsewhere.

The reason why this definition is interesting can be deduced from [7, proposition V.3.4], which gives a description of dualising complexes in the stalk. This description is given by a purity result for Ext-functors, hence considering these rather special residual complexes which are constructed from injective hulls makes sense.

The functor E uses the theory of *Cousin complexes*, and this is based on suitable filtrations of X. For more information, see [7, chapter IV].

This "equivalence" of residual and dualising complexes is one of the subtle points in the proof, and things are not correct the way they are stated. For a discussion of the problems, and a solution, see the discussion around [3, lemma 3.2.1]. This is where the main technical part of the proof is found [7, §VI.2–VI.5]. It requires checking lots and lots of commutative diagrams, and this is one of the reasons for the existence of [3].

4.2 Deligne's proof: go straight for the right adjoint

4.2.1 Introduction

If one wishes to settle for a Grothendieck duality theory, without dualising and residual complexes, it is possible to prove the existence of the right adjoint $f^!$ by other means [4, 17]. One can then show that the remaining aspects of Grothendieck duality follow from the existence of this adjoint.

The idea for this approach comes from Verdier duality, which is a generalised Poincaré duality for topological spaces. It is possible to obtain the results in an almost formal way, if one has a good theory of "cohomology with proper support".

4.2.2 Nagata's compactification theorem

The main idea in Deligne's approach is to replace the morphism $f : X \to Y$ by more tractable ones. As in the case of general topology it is often easier to prove something for compact spaces. The notion of compactness (in the usual sense, often denoted quasicompactness) is only mildly interesting, and does not suffice to prove Grothendieck duality. The correct notion of compactness is properness. So we make the following definition.

Definition 44. Let $f : X \to Y$ be a morphism of schemes. It is *compactifiable* (in the terminology of [7, appendix]) if there exist morphisms $g : X \to \overline{X}$ and $h : \overline{X} \to Y$ such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & \overline{X} \arrow[ur, "h"'] & \end{tikzcd} \tag{4.12}$$

such that

1. g is an open immersion;
2. h is proper.

The question now becomes: which morphisms are compactifiable? The answer is very interesting, and given by the following theorem [15, 14, 2].

Theorem 45 (Nagata's compactification theorem)**.** Let $f : X \to Y$ be separated and of finite type between quasicompact and quasiseperated schemes. Then f is compactifiable.

Hence this suffices to obtain the existence of $f^!$ in a very general context, e.g. noetherian schemes.

Remark that the history of this theorem is intriguing: it was proved in 1962, but in the language of Zariski–Riemann spaces and valuations, which is algebraic geometry in the sense of Zariski and Weil. Hence the proof nor the statement were known in the language of schemes. This has now changed [12, 2, 5]. This result would be an interesting topic for another series of lectures in the seminar.

4.2.3 Outline of the proof

Using this notion of compactifiability he defines a functor $f_!$ (or $\mathbf{R}f_!$, depending on the notation, but it is *not* a derived functor) which is related to $\mathbf{R}f_*$. Then using a result by Verdier [18] we get a right adjoint $f^!$ for $\mathbf{R}f_*$. Then he proves that $f_!$ and $f^!$ are adjoint to eachother (in the context of pro-objects), and deduces some properties. For more deductions on the properties of $f^!$ see [17].

A similar approach is by the way taken to define $f_!$ and $f^!$ in the context of étale cohomology [SGA4$_3$, exposé XVII].

Remark that this approach is completely *orthogonal* to the approach outlined in the previous section:

> We prove the existence of $f^!$ and *afterwards* try to interpret dualising complexes.

4.3 Neeman's proof: Brown's representability theorem

4.3.1 Introduction

The previous two approaches have in common that at some point one takes a factorisation of a morphism into two specific types of morphisms, proceeds to obtain Grothendieck duality for each type separately, and then glues these together by checking compatibility and independence of the choice of factorisation. What if we could get rid of this?

This is done in Neeman's approach [16]: we are looking for a right adjoint to a functor between triangulated categories. We don't have to consider the underlying morphism inducing this functor. By using an abstract result it is possible to obtain the existence of this functor immediately! The reason why this (easy, but formal) approach took so long is that one has to use ideas (or tools) from topology and apply them to study triangulated categories.

4.3.2 Existence of adjoint functors

The general result that we wish to apply is *Brown representability*. The shape it takes in our case is the following:

Theorem 46 (Brown representability)**.** Let $\mathcal{T}$ be a compactly generated triangulated category. Let $H\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ be a homological functor, i.e. it is contravariant and takes triangles to long exact sequences. Suppose furthermore that

$$H\left(\coprod_{\lambda\in\Lambda} t_\lambda\right) \to \prod_{\lambda\in\Lambda} H(t_\lambda) \tag{4.13}$$

is an isomorphism for all small coproducts. Then H is representable.

This representability means we can write H as $\mathrm{Hom}_{\mathcal{T}}(-,t)$ for some $t\in\mathcal{T}$. Given this result, the adjoint functor theorem for compactly generated triangulated categories becomes really easy:

Theorem 47. Let $\mathcal{S}$ be a compactly generated triangulated category, $\mathcal{T}$ be a triangulated category, let $F\colon \mathcal{S}\to\mathcal{T}$ be a triangulated functor, suppose that F respects coproducts, i.e. the natural maps

$$F(s_\lambda) \to F\left(\coprod_{\lambda\in\Lambda} s_\lambda\right) \tag{4.14}$$

turn $F(\coprod_{\lambda\in\Lambda} s_\lambda)$ into a coproduct. Then F admits a right adjoint $G\colon \mathcal{T}\to\mathcal{S}$.

Proof. We consider the homological functor $\mathrm{Hom}_{\mathcal{T}}(F(-),t)$, for each $t\in\mathcal{T}$. This functor is representable by Brown representability, hence we find an object $G(t)\in\mathcal{T}$ such that

$$\mathrm{Hom}_{\mathcal{T}}(F(-),t) \cong \mathrm{Hom}_{\mathcal{S}}(-,G(t)). \tag{4.15}$$

□

To apply this theorem, we need to know that $\mathbf{D}(\mathrm{Qcoh}/X)$ is compactly generated. This is the case if X is quasicompact and separated. Remark that we have removed the bounded below conditions that were so pervasive before. This is *crucial* in this approach to Grothendieck duality! So then we apply this argument to separated morphisms between quasicompact and separated schemes, to obtain Grothendieck duality almost for free by categorical nonsense.

Hence this approach is again completely orthogonal to the original approach by Hartshorne: one looks for a functor $f^!$ and then deduces properties of this functor and the related notion of dualising complexes.

4.4 Murfet's proof: the mock homotopy category of projectives

4.4.1 Introduction

So far we have had a geometric approach and two categorical ones. By using the mock homotopy category of projectives Murfet tries to reconcile the two a little (albeit it still firmly rooted in an abstract language). It is based on several interesting observations.

This approach promises interesting generalisations and new insights, e.g. it is possible to prove Grothendieck duality for sufficiently nice Artin stacks, or look for noncommutative interpretations of Grothendieck duality.

4.4.2 Sketch of the ideas

On the injective side In [10, §6] Henning Krause realises Grothendieck duality at a stage before derived categories by appealing to Neeman's application of Brown representability and the following facts:

1. for X a noetherian separated scheme we can write

   (4.16) $\mathbf{D}(\mathrm{Qcoh}/X) \cong \mathbf{K}(\mathrm{Inj}/X)/\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}/X)$;

2. for X a noetherian separated scheme we have

   (4.17) $\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X) \cong \mathbf{K}(\mathrm{Inj}/X)^{\mathrm{c}}$

   (i.e. the compact objects constitute the derived category of interest).

If one has seen model categories before: this is in the spirit of the injective model structure, we consider the subcategory of fibrant–cofibrant objects as a model for the derived category, and in this explicit form we can do computations. So $f^!$ now lives on the level of $\mathbf{K}(\mathrm{Inj}/X)$ and $\mathbf{K}(\mathrm{Inj}/Y)$.

On the projective side In [9] Peter Jørgensen realises Grothendieck duality at a stage before derived categories by appealing to Neeman's application of Brown representability and the following facts:

1. for X a noetherian *affine* scheme we can write

   (4.18) $\mathbf{D}(\mathrm{Qcoh}/X) \cong \mathbf{K}(\mathrm{Proj}\,/X)/\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}\,/X)$;

2. for X a noetherian *affine* scheme we have

   (4.19) $\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X)^{\mathrm{op}} \cong \mathbf{K}(\mathrm{Proj}\,/X)^{\mathrm{c}}$

   i.e. the compact objects constitute the derived category of interest).

The awfully strict condition in this case immediately brings us to the next observation.

Flat versus projective

1. there are not enough projective objects on a non-affine scheme;
2. the homotopy category of projective objects in the affine case has the following model:

   $$\mathbf{K}(\mathrm{Proj}/X) \cong \mathbf{K}(\mathrm{Flat}/X)/\mathbf{K}_{\mathrm{pac}}(\mathrm{Flat}/X) \tag{4.20}$$

   where the index pac means the "pure acyclic flat complexes", i.e. the ones that are acylic complexes of flat sheaves that remain acyclic after tensoring with a sheaf (the purity can be characterised as having flat kernels for all the differentials).

In the case of a non-affine scheme X we could *mimick* this definition, and set

$$\mathbf{K}_{\mathrm{m}}(\mathrm{Proj}/X) \coloneqq \mathbf{K}(\mathrm{Flat}/X)/\mathbf{K}_{\mathrm{pac}}(\mathrm{Flat}/X). \tag{4.21}$$

This *mock homotopy category of projectives* has all the good properties, similar to the ones of $\mathbf{K}(\mathrm{Inj}/X)$, agrees with the homotopy category of projectives in the affine case and has interesting information in the non-affine case. For more information on the motivation, see [13, chapter 1].

Hence we can now state Grothendieck duality as follows.

Theorem 48 (Grothendieck duality)**.** Let X be a noetherian separated scheme. Let $\mathcal{I}$ be a bounded-below complexe of injective quasicoherent sheaves. Then we have a diagram

$$\begin{tikzcd} \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X)^{\mathrm{op}} \arrow[r, "\mathbf{R}\mathcal{H}om(-,\mathcal{I})"] \arrow[d, hook] & \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}/X) \arrow[d, hook] \\ \mathbf{K}_{\mathrm{mock}}(\mathrm{Proj}/X) \arrow[r, "\mathcal{I}\otimes-"] & \mathbf{K}(\mathrm{Inj}/X) \end{tikzcd} \tag{4.22}$$

where the vertical inclusions are the inclusions of the compact objects. This diagram is well-defined, commutes and has equivalences horizontally if and only if $\mathcal{I}$ is a (pointwise) dualising complex.

Hence we have restated Grothendieck duality in terms of homotopy categories and compact objects.

4.5 Other proofs

4.5.1 Rigid dualising complexes

Based on a generalisation of the notion of dualising complexes to noncommutative algebra [1] by Michel van den Bergh it is possible to introduce the notion of a rigid dualising complex [20], as was done by Amnon Yekutieli and James Zhang. The notion of *rigidity* is encountered in other areas of algebraic geometry as well: the idea is to add more structure to an object, to kill its automorphisms. Examples of these are level structures on elliptic curves, marked points on curves, fixing a special isomorphism (instead of just requiring that there is one), ... They also argue that the cumbersome proofs in [7] are due to a lack of rigidity.

After introducing the notion of a rigid dualising complex the theory goes along similar lines: we treat finite and smooth morphisms separately and apply the same type of reduction to studying residues over $\mathbb{P}^1_k$ [19]. This article seems to be forever in preparation, only a preprint is available.

There is also a noncommutative analogue of this approach.

4.5.2 Pseudo-coherent complexes

One is invited to read [11]. As far as I can tell it is mostly a technical improvement, to put techniques from [SGA6] to good use, and less of a radically new approach. The notes themselves are really nice, and they give a good background to derived categories. I hope to get back to this approach at some point and understand the main difference(s) with other approaches.

4.5.3 More?

Maybe there are more proofs available. But as far as I can tell, these are the main approaches. In the 80ties some work has been done, but these seem to be refinements, not alternative proofs. Feel free to contact me if you have anything to tell.

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