Degeneration in positive characteristic

Jens Hemelaer

April 29, 2015

The main goal of these notes is to prove degeneration of the Hodge-to-de Rham spectral sequence for a proper smooth scheme X over a perfect field k of characteristic $p > 0$. We will need that X is of dimension $< p$ and that X can be lifted to $W_2(k)$, the ring of Witt vectors of length 2.

These notes are mainly based on the well-written text by Illusie [Ill02].

Contents

1 Derived categories

We give a short overview of the theory of derived categories.

We use the cohomological convention for chain complexes, i.e. the differential increases the degree. The category of chain complexes over an abelian category $\mathcal{A}$ will be denoted by $\mathrm{C}(\mathcal{A})$. We say a chain complex L is bounded below (resp. bounded above, resp. bounded) if $L^i = 0$ for sufficiently small i (resp. for sufficiently large i, resp. for i outside of a bounded interval). A chain complex is said to be *concentrated* in a certain range, if it is zero outside of this range. Following [Ill02], we use the 'topological' convention for shifts of chain complexes: we define the chain complex $C[i]$ as $C[i]^n = C^{i+n}$ with differential $\mathrm{d}^n_{C[i]} = (-1)^i \mathrm{d}^{n+i}_C$. If C is concentrated in degree 0, then $C[i]$ is concentrated in degree $-i$.

For a chain complex L, we denote the degree n cocycles by $\mathrm{Z}^n L$, the degree n coboundaries by $\mathrm{B}^n L$ and the homology by $\mathrm{H}^n L$. If $\mathcal{A}$ is the category of $\mathcal{O}_X$-modules associated to a scheme X, we will use the notation $\mathcal{H}^n L$ instead of $\mathrm{H}^n L$ to avoid confusion with sheaf cohomology. In this case, we also use the notation $\mathrm{C}(X)$ instead of $\mathrm{C}(\mathcal{A})$. For a morphism $f : C \to D$ of complexes, there is an induced morphism in homology $\mathrm{H}^n C \to \mathrm{H}^n D$. If this is an isomorphism

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
0 \arrow[u] \arrow[r] & L^{n+1} \arrow[u] \arrow[r] & 0 \arrow[u] \\
\mathrm{Z}^n L \arrow[u] \arrow[r] & L^n \arrow[u] \arrow[r] & L^n \arrow[u] \\
L^{n-1} \arrow[u] \arrow[r] & L^{n-1} \arrow[u] \arrow[r] & L^{n-1} \arrow[u] \\
\cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] \\
\tau_{\leq n} L & L & L^{\leq n}
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdots & \cdots & \cdots \\
L^{n+1} \arrow[u] \arrow[r] & L^{n+1} \arrow[u] \arrow[r] & L^{n+1} \arrow[u] \\
L^n \arrow[u] \arrow[r] & L^n \arrow[u] \arrow[r] & L^n/\mathrm{B}^n L \arrow[u] \\
0 \arrow[u] \arrow[r] & L^{n-1} \arrow[u] \arrow[r] & 0 \arrow[u] \\
\cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] \\
L^{\geq n} & L & \tau_{\geq n} L
\end{tikzcd}$$

for every n, then f is called a *quasi-isomorphism.* For a complex C we can define its cohomology complex

$$\mathrm{H}^\bullet C = \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i C[-i],$$

equipped with zero differential. It's not necessarily true that there is a quasi-isomorphism $C \to \mathrm{H}^\bullet C$ or $\mathrm{H}^\bullet C \to C$. So taking homology forgets important information, and this will be the motivation for derived categories.

We define the *naive truncation* $L^{\leq n}$ (resp. $L^{\geq n}$) as the quotient (resp. subcomplex) of L that coincides with L in degrees $i \leq n$ (resp. $i \geq n$) and that is zero elsewhere. The *canonical truncation* $\tau_{\leq n} L$ (resp. $\tau_{\geq n} L$) is defined to be the subcomplex (resp. quotient) of L that coincides with L in degrees $i < n$ (resp. $i > n$), is given by $\mathrm{Z}^n L$ (resp. $L^n/\mathrm{B}^n L$) in degree n, and is zero elsewhere. The canonical truncation behaves more naturally with respect to taking homology: the maps $\tau_{\leq n} L \to L$ (resp. $L \to \tau_{\geq n} L$) are isomorphisms in homology in degrees $i \leq n$ (resp. degrees $i \geq n$).

Definition 1.1. Let $\mathcal{A}$ be an abelian category. The *derived category* $\mathrm{D}(\mathcal{A})$ is the localization of $\mathrm{C}(\mathcal{A})$ with respect to quasi-isomorphisms. More precisely, there is a map $\mathrm{C}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ that is universal among functors sending quasi-isomorphisms to isomorphisms, i.e. this map sends quasi-isomorphisms to isomorphisms and for any other category $\mathcal{C}$ and functor $\mathrm{C}(\mathcal{A}) \to \mathcal{C}$, there is a factorization

$$\begin{tikzcd}
\mathrm{C}(\mathcal{A}) \arrow[r, "F"] \arrow[d] & \mathcal{C} \\
\mathrm{D}(\mathcal{A}) \arrow[ur, dotted] &
\end{tikzcd}$$

through $\mathrm{D}(\mathcal{A})$.

Note that it is not trivial that such a category $\mathrm{D}(\mathcal{A})$ exists, but existence will follow from the explicit construction we are going to give. For this explicit construction, we need to pass through the *homotopy category* $\mathrm{K}(\mathcal{A})$.

Definition 1.2. Let $f, g : C \to D$ be two maps of chain complexes. Then f and g are *homotopic* if there exists a family of maps $s^n : C^n \to D^{n+1}$ such that

$$s^{n-1} \circ \mathrm{d}_C^n + \mathrm{d}_D^{n+1} \circ s^n = f - g.$$

The *homotopy category* $\mathrm{K}(\mathcal{A})$ is then the category with the same objects as $\mathrm{C}(\mathcal{A})$ and as arrows homotopy classes of maps in $\mathrm{C}(\mathcal{A})$.

We mention the following theorem from [Kel07].

Theorem 1.3. *The derived category* $\mathrm{D}(\mathcal{A})$ *can be constructed as follows:*

- *The objects are the same as in* $\mathrm{C}(\mathcal{A})$*.*
- *The arrows from* L *to* M *are triples* (M', f, s) *with* M' *another object,* f *a map* $L \to M'$ *and* s *a quasi-isomorphism* $M \to M'$*. We use the suggestive notation* $s^{-1}f$ *to denote this map.*

$$\begin{tikzcd} & M' & \\ L \arrow[ur, "f"] & & M \arrow[ul, "s"'] \end{tikzcd}$$

- *Two arrows* (M', f, s) *and* (M'', f', s') *are the same if there exists some arrow* (M''', f'', s'') *and maps* $M' \to M'''$, $M'' \to M'''$ *such that the following diagram commutes in* $\mathrm{K}(\mathcal{A})$*:*

$$\begin{tikzcd} & M' \arrow[d] & \\ L \arrow[ur, "f"] \arrow[r, "f''"] \arrow[dr, "f'"'] & M''' & M \arrow[ul, "s"'] \arrow[l, "s''"'] \arrow[dl, "s'"] \\ & M'' \arrow[u] & \end{tikzcd}.$$

- *For every two arrows* $(M', f, s) : L \to M$ *and* $(N', g, t) : M \to N$*, there is an arrow* $(N'', g', s') : M' \to N'$*. We define the composition of* (M', f, s) *and* (N', g, t) *to be the triple* $(N'', g'f, s't)$*. The composition is associative.*

$$\begin{tikzcd} & & N'' & & \\ & M' \arrow[ur, "g'"] & & N' \arrow[ul, "s'"'] & \\ L \arrow[ur, "f"] & & M \arrow[ul, "s"'] \arrow[ur, "g"] & & N \arrow[ul, "t"'] \end{tikzcd}$$

- *The maps* $(L, \mathrm{id}, \mathrm{id}) : L \to L$ *are identity maps.*

Remark 1.4. There is an analogous (and of course equivalent) construction of the derived category with as maps $L \to M$ triples (M', s, f) where s is a quasi-isomorphism $M' \to L$ and f is a map $M' \to M$.

$$\begin{tikzcd} & M' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ L & & M \end{tikzcd}$$

Definition 1.5. We call two chain complexes *quasi-isomorphic* if they become isomorphic in the derived category. We call a chain complex *decomposable* (or *formal*) if it is quasi-isomorphic to its cohomology. For a decomposable chain complex C, a *decomposition* is a map $C \to \mathrm{H}^\bullet C$ in the derived category, inducing *the identity* in homology.

Remark 1.6. If a chain complex C is indecomposable, there are in general many different decompositions.

In [Tho01], an example is given of a chain complex that is not decomposable:

Example 1.7. The two chain complexes of $\mathbb{C}[x, y]$-modules

$$\mathbb{C}[x, y]^{\oplus 2} \xrightarrow{(x,y)} \mathbb{C}[x, y] \qquad \text{and} \qquad \mathbb{C}[x, y] \xrightarrow{0} \mathbb{C},$$

have the same homology groups, but are not quasi-isomorphic. As a consequence, the first chain complex is not decomposable.

Note that the category of $\mathbb{C}[x, y]$-modules has global dimension 2. It is not possible to give a similar example if $\mathcal{A}$ has global dimension 0 or 1. For a proof we refer to [Kel07], section 2.5.

Definition 1.8. We denote by $\mathrm{D}^+(\mathcal{A})$ (resp. $\mathrm{D}^-(\mathcal{A})$, resp. $\mathrm{D}^b(\mathcal{A})$) the full subcategory of $\mathrm{D}(\mathcal{A})$ consisting of these complexes for which the cohomology complex is bounded below (resp. bounded above, resp. bounded). We use analogous notations for $\mathrm{K}(\mathcal{A})$.

The following proposition [Sta15, Tag 05TA] tells us how derived functors give functors between derived categories, and how to compute them.

Proposition 1.9. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories.*

- *If every object of $\mathcal{A}$ injects into some F-acyclic object, then $\mathbf{R}F$ is defined on all of $\mathrm{K}^+(\mathcal{A})$ and we obtain an exact functor*

  $$\mathbf{R}F : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B}).$$

  Moreover, let C be a chain complex over $\mathcal{A}$. Then any isomorphism in $\mathrm{D}(\mathcal{A})$ to a bounded below complex D with F-acyclic components, yields an isomorphism

  $$\mathbf{R}F(C) \to F(D).$$

- *If every object of $\mathcal{A}$ is a quotient of an F-acyclic object, then $\mathbf{L}F$ is defined on all of $\mathrm{K}^-(\mathcal{A})$ and we obtain an exact functor*

  $$\mathbf{L}F : \mathrm{D}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{B}).$$

  Moreover, let C be a chain complex over $\mathcal{A}$. Then any isomorphism in $\mathrm{D}(\mathcal{A})$ to a bounded above complex D with F-acyclic components, yields an isomorphism

  $$\mathbf{L}F(C) \to F(D).$$

Remark 1.10. An *exact functor* between derived categories is, informally, a functor sending long exact sequences to long exact sequences. The precise formulation uses the so-called triangulated structure on a derived category, which is beyond the scope of these notes. The proposition above is, of course, still valid after replacing 'exact functor' with just 'functor'.

Remark 1.11. Note that all injective objects are F-acyclic for left exact functors. Similarly, all projective objects are F-acyclic for right exact functors.

Proposition 1.12. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives. Let K be a complex in $\mathrm{D}^+(\mathcal{A})$. Then there is a spectral sequence*

$$E_1^{pq} = \mathrm{R}^q F(K^p) \Rightarrow \mathrm{R}^{p+q} F(K),$$

called the first spectral sequence of hypercohomology*.*

Our main example occurs when we take $K = \Omega^\bullet_{X/k}$ and $F = f_*$, where X is a scheme over a field k with structure map $f : X \to \operatorname{Spec} k$. This gives the spectral sequence

$$E_1^{pq} = \mathrm{R}^q f_*(\Omega^p_{X/k}) \Rightarrow \mathrm{R}^{p+q} f_*(\Omega^\bullet_{X/k}).$$

Taking global sections gives an exact equivalence of abelian categories between the category of $\mathcal{O}_{\operatorname{Spec} k}$-modules and k-vector spaces. So we can take global sections at each side of the spectral sequence. We then get

$$E_1^{pq} = \mathrm{H}^q(X, \Omega^p_{X/k}) \Rightarrow \mathrm{H}^{p+q}(X, \Omega^\bullet_{X/k}) = \mathrm{H}^{p+q}_{\mathrm{dR}}(X/k),$$

i.e. de Hodge-to-de Rham spectral sequence. Note that if X is proper over k, then this already shows that the de Rham cohomology groups are finite-dimensional: the $\Omega^p_{X/k}$ are coherent, so each term in E^1 is finite-dimensional, and from this it follows that also the groups to which they converge are finite-dimensional (being finite-dimensional is closed under taking subquotients and extensions). Another fact we can already deduce from the existence of the spectral sequence is

$$\sum_{i+j=n} \dim_k \mathrm{H}^j(X, \Omega^i_{X/k}) \geq \dim_k \mathrm{H}^n_{\mathrm{dR}}(X/k)$$

This is trivial after realizing that taking homology always reduces the dimension. Moreover, it *strictly* reduces the dimension whenever some differential is non-zero. So the degeneration of the Hodge-to-de Rham spectral sequence is actually equivalent to the equality

$$\sum_{i+j=n} \dim_k \mathrm{H}^j(X, \Omega^i_{X/k}) = \dim_k \mathrm{H}^n_{\mathrm{dR}}(X/k).$$

2 Degeneration in positive characteristic

2.1 Step A: Everything lifts globally

Recall the setting from last time: S is some scheme over $\mathbb{F}_p$, T a flat lift of S to $\mathbb{Z}/p^2\mathbb{Z}$, and X a smooth S-scheme. Denote the Frobenius twist of X by X', and assume that X,X' and the relative Frobenius $F : X \to X'$ can be lifted (as a whole) to $G : Z \to Z'$ over T.

$$\begin{tikzcd}
X \arrow[rr, "F"] \arrow[dr] & & X' \arrow[dl] & Z \arrow[rr, "G"] \arrow[dr] & & Z' \arrow[dl] \\
& S \arrow[rrr] \arrow[d] & & & T \arrow[d] & \\
& \operatorname{Spec} \mathbb{F}_p \arrow[rrr] & & & \operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z} &
\end{tikzcd}$$

Note that we have an exact sequence of $\mathbb{Z}/p^2\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0.$$

Pulling back this exact sequence to T gives

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{p} \mathcal{O}_T \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

which is again exact by flatness of T (note that S and T have the same underlying topological space). We have completely analogous short exact sequences for Z and Z' (which are flat over T, so in particular flat over $\operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z}$).

Because X (so also X') is smooth by assumption, the cotangent sheaf $\Omega^1_{X/S}$ is locally free, in particular flat. Moreover, we saw in the second lecture [Pre15, Proposition 2.4 (2)] that

$$\Omega^1_{Z/T} \otimes_{\mathcal{O}_Z} \mathcal{O}_X \simeq \Omega^1_{X/S}.$$

So we get a short exact sequence

$$0 \longrightarrow \Omega^1_{X/S} \xrightarrow{p} \Omega^1_{Z/T} \longrightarrow \Omega^1_{X/S} \longrightarrow 0.$$

Proposition 2.1. *Suppose we are in the above situation. Then:*

a) Multiplication by p induces an isomorphism

$$\Omega^1_{X/S} \xrightarrow[p]{\cong} p\Omega^1_{Z/T} \ .$$

b) Pulling back differentials on Z' to Z gives multiples of p, i.e. there is a factorization

$$\begin{tikzcd}
\Omega^1_{Z'/T} \arrow[r, "G^*"] \arrow[dr, dashed] & G_*\Omega^1_{Z/T} \\
& pG_*\Omega^1_{Z/T} \arrow[u, hook]
\end{tikzcd} .$$

c) There is a factorization

$$\begin{tikzcd}
\Omega^1_{Z'/T} \arrow[r, "G^*"] \arrow[d] & pG_*\Omega^1_{Z/T} \\
\Omega^1_{X'/S} \arrow[r, dashed, "\text{“}G^*/p\text{”}"] & F_*\Omega^1_{X/S} \arrow[u, "\cong"', "p"]
\end{tikzcd} .$$

Moreover, the image of G^*/p is contained in the kernel of the differential of the de Rham complex, i.e. in $\mathrm{Z}^1 F_*\Omega^\bullet_{X/S}$, and the composite with the projection to $\mathcal{H}^1 F_*\Omega^\bullet_{X/S}$ gives the Cartier isomorphism

$$\Omega^1_{X'/S} \xrightarrow[\cong]{C^{-1}} \mathcal{H}^1 F_*\Omega^\bullet_{X/S}$$

$$\mathrm{d}s \longmapsto s^{p-1}\,\mathrm{d}s$$

Proof. a) This follows from the exact sequence we already found.

b) Because G is a lift of F, we have for a local section of $\mathcal{O}_{Z'}$

$$G^*(a) = a^p + pb \text{ for some } b$$
$$G^*(\mathrm{d}a) = pa^{p-1}\,\mathrm{d}a + p\,\mathrm{d}b.$$

c) We have $\Omega^1_{X'/S} \simeq \Omega^1_{Z'/T} \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_{X'}$, so the factorization follows by the universal property of the pullback (note that $F_*\Omega^1_{X/S}$ is an $\mathcal{O}_{X'/S}$-module). Moreover, we have for a local section a of $\mathcal{O}_{Z'}$

$$(G^*/p)(\mathrm{d}a) = a^{p-1}\,\mathrm{d}a + \mathrm{d}b,$$

which is a cocycle and actually the Cartier iso

$$C^{-1}(\mathrm{d}a) = a^{p-1}\,\mathrm{d}a$$

after quotienting out the coboundaries.

□

Corollary 2.2. *Let*

$$\varphi_G : \bigoplus_{i\in\mathbb{Z}} \Omega^i_{X'/S}[-i] \to F_*\Omega^\bullet_{X/S}$$

be defined by

$$\begin{aligned} F^* &: \mathcal{O}'_X \to F_*\mathcal{O}_X && \textit{in degree 0,} \\ G^*/p &: \Omega^1_{X'/S} \to F_*\Omega^1_{X/S} && \textit{in degree 1,} \end{aligned}$$

and then extended multiplicatively, i.e. via

$$\Omega^i_{X'/S} = \bigwedge^i \Omega^1_{X'/S} \xrightarrow{\bigwedge^i (G^*/p)} \bigwedge^i F_*\Omega^1_{X/S} \xrightarrow{\textit{mult.}} F_*\Omega^i_{X/S}\ .$$

Then φ_G is a quasi-isomorphism, inducing the Cartier isomorphism C^{-1} in homology.

Proof. It is enough to prove the statement in degrees 0 and 1, because the Cartier isomorphism is in higher degrees also determined by multiplicativity. But the statement in degree 0 follows directly from the definition of the Cartier isomorphism, and the statement in degree 1 is Proposition 2.1, part c. □

2.2 Step B: The relative Frobenius lifts only locally; decomposability in degrees 0 and 1

This is the trickier part. We do not longer assume that *everything* lifts to T, we only assume that X' lifts to T (to some fixed Z'). Note that everything lifts at least locally: the obstruction to lifting X lies in $\mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X)$, and the obstruction to lifting the Frobenius lies in $\mathrm{Ext}^1(F^*\Omega^1_{X'}, \mathcal{O}_X)$. So the existence of local liftings follows from the fact that X and X' are smooth (note that X' is a pullback of X, so if X is smooth, then X' is automatically also smooth). We will prove that $\tau_{\leq 1}F_*\Omega^\bullet_{X/S}$ is still decomposable. To this extent, we will try to "glue" quasi-isomorphisms like in the previous step. First we need the following uniqueness statement.

Lemma 2.3. *To any pair*

$$\begin{tikzcd} Z_1 \arrow[dr, "G_1"] & \\ & Z' \\ Z_2 \arrow[ur, "G_2"] & \end{tikzcd}$$

of liftings of F, is associated canonically a map

$$h(G_1, G_2) : \Omega^1_{X'/S} \to F_*\mathcal{O}_X$$

such that

$$\varphi^1_{G_2} - \varphi^1_{G_1} = \mathrm{d}h(G_1, G_2).$$

If $G_3 : Z_3 \to Z'$ is a third lifting of F, one has

$$h(G_1, G_2) + h(G_2, G_3) = h(G_1, G_3).$$

Proof. First suppose Z_1 and Z_2 are isomorphic as deformations, i.e. there is an isomorphism

$$\begin{tikzcd} Z_1 \arrow[rr, "\cong"] \arrow[dr] & & Z_2 \arrow[dl] \\ & T & \end{tikzcd}$$

inducing the identity on X. Choose such an isomorphism $u : Z_1 \to Z_2$. Then both G_2u and G_1 lift F. Therefore they differ by some

$$h_u : \Omega^1_{X'/S} \to F_*\mathcal{O}_X.$$

Indeed, consider the lifting diagram

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d] & Z' \arrow[d] \\ Z_1 \arrow[r] \arrow[ur, shift left, "G_1"] \arrow[ur, shift right, "G_2u"'] & T \end{tikzcd}$$

and notice that Z_1 is a first-order infinitesimal thickening with ideal $\mathcal{O}_X$. So $G_2u - G_1$ is given by some map

$$\Omega^1_{Z'/T} \to F_*\mathcal{O}_X,$$

or equivalently, a map

$$\Omega^1_{X'/S} \to F_*\mathcal{O}_X.$$

Now if we take another isomorphism $v : Z_1 \to Z_2$, we have a similar lifting diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Z_2 \arrow[d] \\
Z_1 \arrow[r] \arrow[ur, shift left, "u"] \arrow[ur, shift right, "v"'] & T
\end{tikzcd},$$

and now $v - u$ is given by some map $\Omega^1_{X/S} \to \mathcal{O}_X$. However, after composing with G_2 we get

$$\begin{tikzcd}
X \arrow[rr, "F"] \arrow[dd] \arrow[dr] & & Z' \arrow[dd] \\
& Z_2 \arrow[ur, "G_2"] \arrow[dr] & \\
Z_1 \arrow[rr] \arrow[ur, shift left, "u"] \arrow[ur, shift right, "v"'] & & T
\end{tikzcd}$$

Note that postcomposing with G_2 sends derivations to derivations functorially, so using Yoneda lemma we translate the above to

$$\begin{tikzcd}
F^*\Omega^1_{X'/S} \simeq F^*\Omega^1_{Z'/T} \arrow[r, "G_2u - G_2v"] \arrow[d, "F^*"'] & \mathcal{O}_X \\
\Omega^1_{X/S} \arrow[ur] &
\end{tikzcd}.$$

Because we are in characteristic p, we have for any local section a in $\mathcal{O}_{X'}$ that

$$F^*(\mathrm{d}a) = \mathrm{d}(a^p) = pa^{p-1}\,\mathrm{d}a = 0.$$

So in particular, $G_2u = G_2v$, which means the choice of isomorphism does not matter here! As a consequence, we can just set $h(G_1, G_2) = h_u$ for some choice of isomorphism u. For a local section a of $\mathcal{O}_{Z'}$ we have the following explicit descriptions of G_1^*, G_2^* and $h(G_1, G_2)^*$:

$$\begin{aligned}
G_1^*(a) &= a^p + pb_1 \\
G_2^*(a) &= a^p + pb_2 \\
h(G_1, G_2)^*(a) &= b_2 - b_1
\end{aligned}$$

(notice that $h(G_1, G_2)^*$ maps to $\mathcal{O}_X$ and that the inclusion $\mathcal{O}_X \subseteq \mathcal{O}_Z$ is given by multiplication by p, so $h(G_1, G_2)$ is the difference of G_1 and G_2, although it might not look like it). For the φ^1's we get similarly

$$(\varphi^1_{G_2} - \varphi^1_{G_1})(\mathrm{d}a) = d(b_2 - b_1) = \mathrm{d}h(G_1, G_2)(a).$$

The formula

$$h(G_1, G_2) + h(G_2, G_3) = h(G_1, G_3)$$

is now also immediate. $\square$

Now let $X = \bigcup_i U_i$ be an open covering, and choose liftings

$$\begin{tikzcd}
U_i \arrow[rr, "F_i"] \arrow[dr] & & X' \arrow[dl] & Z_i \arrow[rr, "G_i"] \arrow[dr] & & Z' \arrow[dl] \\
& S \arrow[rrr] & & & T &
\end{tikzcd}.$$

For each i, consider the map

$$\Omega^1_{X'/S}[-1]\Big|_{U_i} \xrightarrow{f_i = \varphi^1_{G_i}} F_* \Omega^\bullet_{X/S}\Big|_{U_i}$$

and for each pair (i, j) the maps

$$\Omega^1_{X'/S}[-1]\Big|_{U_{ij}} \xrightarrow{h_{ij}} F_* \Omega^*_{X/S}\Big|_{U_{ij}}$$

with

$$h_{ij} = h(G_i\big|_{U_{ij}}, G_j\big|_{U_{ij}}).$$

So we have

$$f_j - f_i = \mathrm{d}h_{ij} \qquad \text{on } U_{ij} \tag{1}$$

$$h_{ij} + h_{jk} = h_{ik} \qquad \text{on } U_{ijk}. \tag{2}$$

Proposition 2.4. *The above data canonically induce a map*

$$\Omega^1_{X'/S}[-1] \longrightarrow \check{\mathcal{C}}(\mathcal{U}, F_* \Omega^\bullet_{X/S}).^1.$$

Proof. The Čech complex on the right is the complex that is in degree n given by

$$\check{\mathcal{C}}(\mathcal{U}, F_* \Omega^\bullet_{X/S})^n = \bigoplus_{a+b=n} \check{\mathcal{C}}^b(\mathcal{U}, F_* \Omega^a_{X/S})$$

and with differential $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2$, where d_1 is the de Rham differential and d_2 the usual Čech complex differential. Recall that

$$\check{\mathcal{C}}^b(\mathcal{U}, \mathcal{F}) = \prod_{i_0 \ldots i_b} \mathcal{F}(U_{i_0} \cup \cdots \cup U_{i_b})$$

and that the differential is given by

$$\mathrm{d}_2 : \check{\mathcal{C}}^b(\mathcal{U}, \mathcal{F}) = \check{\mathcal{C}}^{b+1}(\mathcal{U}, \mathcal{F})$$

$$\mathrm{d}_2(a)_{i_0 \ldots i_{b+1}} = \sum_{k=0}^{p+1} (-1)^k a_{i_0 \ldots \hat{i}_k \ldots i_b}.$$

[1] For a guide to typesetting Čech checks in LaTeX, see `https://pbelmans.wordpress.com/2014/11/24/nitpicking-cech-cohomology/`.

The homomorphism in the proposition should of course map into the degree 1 elements of the Čech complex, being

$$\check{\mathcal{C}}(\mathcal{U}, F_*\Omega^\bullet_{X/S})^1 = \check{\mathcal{C}}^1(\mathcal{U}, F_*\mathcal{O}_X) \oplus \check{\mathcal{C}}^0(\mathcal{U}, F_*\Omega^1_{X/S}).$$

We define it as

$$\begin{array}{ccc} \Omega^1_{X'/S}[-1] & \longrightarrow & \check{\mathcal{C}}^1(\mathcal{U}, F_*\mathcal{O}_X) \oplus \check{\mathcal{C}}^0(\mathcal{U}, F_*\Omega^1_{X/S}) \\ \omega & \longmapsto & \{h_{ij}(\omega)\}_{ij} - \{f_i(\omega)\}_i \end{array}.$$

We need to show that this yields a morphism of complexes, so ω maps to a cocycle for $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2$. This gives:

$$\begin{aligned} &h_{jk} - h_{ik} + h_{ij} = 0 \\ &\mathrm{d}h_{ij} - f_j + f_i = 0 \\ &\mathrm{d}f_i = 0. \end{aligned}$$

The last condition just says that the f_i are maps of complexes, which is part of Corollary 2.2). The first two conditions are the equations 1 and 2 above. $\square$

We have a quasi-isomorphism

$$\epsilon : F_*\Omega^\bullet_{X/S} \to \check{\mathcal{C}}(\mathcal{U}, F_*\Omega^\bullet_{X/S})$$

because the Čech complex defines a resolution. Now define

$$\varphi^1_{Z'} : \Omega^1_{X'/S}[-1] \to F_*\Omega^\bullet_{X/S}$$

as the map in $\mathrm{D}(X')$ given by the triangle

$$\begin{tikzcd} & \check{\mathcal{C}}(\mathcal{U}, F_*\Omega^\bullet_{X/S}) & \\ \Omega^1_{X'/S}[-1] \arrow[ur] & & F_*\Omega^\bullet_{X/S} \arrow[ul, "\epsilon"'] \end{tikzcd}$$

This is independent from the covering $\mathcal{U}$: take another covering $\mathcal{V}$ and set $\mathcal{W} = \mathcal{U} \sqcup \mathcal{V}$. Then we have a diagram

$$\begin{tikzcd} & \check{\mathcal{C}}(\mathcal{U}, F_*\Omega^\bullet_{X/S}) & \\ \Omega^1_{X'/S}[-1] \arrow[ur] \arrow[r] \arrow[dr] & \check{\mathcal{C}}(\mathcal{W}, F_*\Omega^\bullet_{X/S}) \arrow[u] \arrow[d] & F_*\Omega^\bullet_{X/S} \arrow[ul] \arrow[l] \arrow[dl] \\ & \check{\mathcal{C}}(\mathcal{V}, F_*\Omega^\bullet_{X/S}) & \end{tikzcd}$$

that is commutative in $\mathrm{K}(X)$ (even in $\mathrm{C}(X)$). Moreover, it is easy to see that $\varphi^1_{Z'}$ induces C^{-1} on $\mathcal{H}^1$.

2.3 Step C: The relative Frobenius lifts only locally; extending the decomposition to higher degrees

Now we want to construct maps $\varphi^i_{Z'}$ for $2 \le i < p$, extending $\varphi^1_{Z'}$ from the previous subsection. We have

$$(\Omega^1_{X'/S}[-1])^{\overset{\mathrm{L}}{\otimes} i} \to (F_*\Omega^\bullet_{X/S})^{\overset{\mathrm{L}}{\otimes} i}.$$

Because $\Omega^1_{X'/S}$ is locally free, we have

$$(\Omega^1_{X'/S}[-1])^{\overset{\mathrm{L}}{\otimes} i} \simeq (\Omega^1_{X'/S})^{\otimes i}[-i].$$

Similarly,

$$(F_*\Omega^\bullet_{X/S})^{\overset{\mathrm{L}}{\otimes} i} \simeq (F_*\Omega^\bullet_{X/S})^{\otimes i}.$$

Now define

$$\begin{array}{cccc} \Omega^i_{X'/S}[-i] & \longrightarrow (\Omega^1_{X'/S}[-1])^{\otimes i} & \xrightarrow{(\varphi^1_{Z'})^{\otimes i}} (F_*\Omega^\bullet_{X/S})^{\otimes i} & \xrightarrow{\text{mult.}} F_*\Omega^\bullet_{X/S} \\ \omega_1 \wedge \cdots \wedge \omega_i & \longmapsto \frac{1}{i!}\sum_{\sigma\in S_i} \operatorname{sgn}(\sigma)\, \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(i)} & & \end{array}$$

Note that here $\frac{1}{i!}$ exists because of the assumption $i < p$. The above map again induces the Cartier isomorphism C^{-1} in homology, by multiplicativity.

2.4 Witt vectors of length two

We now proved decomposability of $\tau_{<p}F_*\Omega^\bullet_{X/S}$ in the case that X' lifts to T. This will enable us to prove degeneration of the Hodge-to-de Rham spectral sequence in the following case.

Let X be a scheme over a perfect field k of characteristic p. We will consider lifts of X (and its relative Frobenius) over some first-order infinitesimal thickening T of k. One important such infinitesimal thickening is the affine scheme associated to the so-called ring of Witt vectors of length two, $W_2(k)$. As a set, $W_2(k)$ consists of pairs $(a_1, a_2) \in k \times k$. The ring structure is defined by

$$\begin{aligned} (a_1, a_2) + (b_1, b_2) &= (a_1 + b_1, S_2(a, b)), \\ (a_1, a_2)(b_1, b_2) &= (a_1 b_1, P_2(a, b)), \end{aligned}$$

where

$$\begin{aligned} S_2(a, b) &= a_2 + b_2 + p^{-1}(a_1^{p-1} + b_1^{p-1} - (a_1 + b_1)^p), \\ P_2(a, b) &= b_1^p a_2 + b_2 a_1^p. \end{aligned}$$

There is a cartesian diagram

$$\begin{array}{ccc} \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} W_2(k) \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{F}_p & \longrightarrow & \operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z} \end{array}$$

where the map $W_2(k) \to k$ is given by projecting onto the first factor. Moreover, $W_2(k)$ is flat over $\mathbb{Z}/p^2\mathbb{Z}$, i.e. it lifts k.

Now suppose that the scheme X can indeed be lifted globally to a scheme Z over $W_2(k)$. Note that X' is isomorphic to X: it is the pullback of X along the absolute Frobenius of k, which is an isomorphism. So also X' can be lifted to some Z' over $W_2(k)$.

We already proved the following proposition.

Proposition 2.5. *Let k be a perfect field of characteristic p, and X a smooth scheme over $S = \operatorname{Spec} k$. If X is lifted over $T = \operatorname{Spec} W_2(k)$, then $\tau_{<p} F_* \Omega^\bullet_{X/S}$ is decomposable in $\mathrm{D}(X')$. Moreover, if X is of dimension $< p$, then $F_*\Omega^\bullet_{X/S}$ is decomposable.*

In general the relative Frobenius can only be lifted locally. If, in this setting, the relative Frobenius can be lifted globally, then we call X an *ordinary* variety. For example, affine varieties are ordinary.

We now arrive at the main goal of these notes.

Theorem 2.6. *Let k be a perfect field of characteristic p, and X a smooth and proper scheme over $S = \operatorname{Spec} k$, of dimension $< p$. If X is lifted over $T = \operatorname{Spec} W_2(k)$, then the Hodge-to-de Rham spectral sequence*

$$E_1^{ij} = \mathrm{H}^j(X, \Omega^i_{X/k}) \Rightarrow \mathrm{H}^\bullet_{\mathrm{dR}}(X/k)$$

degenerates at E_1.

Proof. Consider the pullback diagram

$$\begin{tikzcd} X \arrow[dr, "F"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right] & & \\ & X' \arrow[r, "\tilde{F}_S"] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "F_S"] & S \end{tikzcd}$$

from the definition of the relative Frobenius F. Note that the absolute Frobenius F_S is an isomorphism, and so also $\tilde{F}_S$ is an isomorphism. In particular they are both flat. We get an isomorphism

$$F_S^* \mathrm{H}^j(X, \Omega^i_{X/k}) \xrightarrow{\sim} \mathrm{H}^j(X', \Omega^i_{X'/k})$$

(taking cohomology commutes with flat base change). Note that we again use the formula $\tilde{F}_S^* \Omega^i_{X'/k} \simeq \Omega^i_{X/k}$ from [Pre15, Proposition 2.4 (2)]. In particular, comparing dimensions gives

$$\dim_k H^j(X, \Omega^i_{X/k}) = \dim_k H^j(X', \Omega^i_{X'/k}).$$

Also, the relative Frobenius F is a homeomorphism, so we have an isomorphism

$$\mathrm{H}^n(X', F_*\Omega^\bullet_{X/k}) \xrightarrow{\sim} \mathrm{H}^n(X, \Omega^\bullet_{X/k}) = \mathrm{H}^n_{\mathrm{dR}}(X/k). \ .$$

Now take a decomposition

$$\varphi : \bigoplus_i \Omega^i_{X'/S}[-i] \xrightarrow{\sim} F_* \Omega^\bullet_{X/S}$$

in $\mathrm{D}(X')$. Taking nth cohomology on both sides yields an isomorphism

$$\bigoplus_{i+j=n} \mathrm{H}^j(X', \Omega^i_{X'/k}) \xrightarrow{\sim} \mathrm{H}^n(X', F_* \Omega^\bullet_{X/k}) \cong \mathrm{H}^n_{\mathrm{dR}}(X/k) \ ,$$

and comparing dimensions gives

$$\dim_k \mathrm{H}^n_{\mathrm{dR}}(X/k) = \sum_{i+j=n} \dim_k \mathrm{H}^j(X, \Omega^i_{X/k}).$$

At the end of Section 1 we showed that this is equivalent to degeneration of the Hodge-to-de Rham spectral sequence at E_1. $\square$

References

[Ill02] Luc Illusie, *Frobenius and Hodge degeneration*, Introduction to Hodge theory, SMF/AMS Texts and Monographs, vol. 8, American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2002, pp. 99–149.

[Kel07] Bernhard Keller, *Derived categories and tilting*, Handbook of tilting theory, London Math. Soc. Lecture Note Ser., vol. 332, Cambridge Univ. Press, Cambridge, 2007, pp. 49–104.

[Pre15] Dennis Presotto, *Differentials and smoothness*, `http://win.ua.ac.be/~pbelmans/seminar/hdr/kahler.pdf`, 2015.

[Sta15] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2015.

[Tho01] R. P. Thomas, *Derived categories for the working mathematician*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), AMS/IP Stud. Adv. Math., vol. 23, Amer. Math. Soc., Providence, RI, 2001, pp. 349–361.