Limits of schemes and density theorems

and counterexamples to degeneration in positive characteristic

Sebastian Klein

1 Introduction

The main aim of this talk is to introduce technology that will help us prove the Hodge-to-deRham degeneration theorem (*H2dR*) in characteristic 0. The results that I will present are rather technical in nature, so I will refrain from giving full proofs in order to not lose track of the general picture.

Before we dive into this material, let us wrap up last time's discussion of H2dR in positive characteristic by giving some examples where the spectral sequences does not degenerate

2 Counterexamples to H2dR in positive characteristic

As we saw last time, H2dR holds for a smooth and proper scheme X over a perfect field k with $p > \dim(X)$ that can be lifted to the Witt vectors of length 2 $\mathrm{W}_2(k)$. There are, however, counterexamples that are not too pathological: for instance, Lang classifies quasi-elliptic surfaces in characteristic 3 in [Lan79]. Along the way, he computes the numbers $h^{i,j}, h^k_{\mathrm{DR}}$ for these surfaces for low i, j, k and comes across an example of a surface X where $h^{1,0}(X) = h^{0,1}(X) = 2$ but $h^1_{\mathrm{DR}}(X) = 3$. This means that H2dR cannot hold for X.

A different strategy to obtain counterexamples is the following: let k be a field of positive characteristic and X an algebraic variety over k. The E_1-page of the Hodge-to-de-Rham spectral sequence is given as

$$E_1^{p,q} = \mathrm{H}^q(X, \Omega^p_{X/k}) .$$

In particular, it contains the map

$$E_1^{1,0} = \mathrm{H}^0(X, \Omega^1_{X/k}) \xrightarrow{\mathrm{d}} \mathrm{H}^0(X, \Omega^2_{X/k}) = E_1^{2,0}$$

given by the exterior derivative. In order to prove that H2dR does not hold for X, it suffices to show that $\mathrm{d} \neq 0$, i.e. we need to exhibit a regular differential on X that is not closed. This strategy is for example used by Mumford (see [Mum61]) or Fossum (see [Fos81]).

Here is a very rough sketch of Mumford's construction: he begins by showing that for every non-singular algebraic surface X over k and not necessarily regular differential ω on X, there exists a surface $\varphi : X^* \to X$ that is regular and separable over X such that $\varphi^*(\omega)$ is a regular differential of X^*. Then, as φ^* is injective in this case and satisfies $\mathrm{d}\varphi^*(\omega) = \varphi^*(\mathrm{d}\omega)$, it suffices to exhibit a non-singular algebraic surface X with a differential ω on X such that $\mathrm{d}\omega \neq 0$. Take $X = \mathbb{P}^2_k$ and $\omega = x\mathrm{d}y$.

2.1 Remark. As mentioned in the last talk, the failure of H2dR for these surfaces shows that they cannot admit a lifting to the Witt vectors of length 2.

3 Limits of schemes

Let us now focus on the tools that we need to prove H2dR in characteristic 0. In this part of the lecture, our goal will be to answer questions along the following lines: if X is a nice scheme over S, and S is given as the projective limit of a system of nice schemes $\varprojlim S_i$, then does there exist some i and a nice scheme X_i over S_i such that $X = X_i \times_{S_i} S$? Furthermore, if $f : X \to Y$ is a nice morphism of S-schemes, then does there exist a j and a nice morphism $f_j : X_j \to Y_j$ of S_j-schemes, such that $f = f_j \times_{S_j} \mathrm{id}_S$?

In order to tackle these questions, let us briefly discuss some generalities about projective limits of schemes:

3.1 Theorem. *Let $(Y_\alpha, \alpha \in I)$ be an projective system of schemes over a directed poset I such that the transition maps $u_{\alpha\beta} : Y_\alpha \to Y_\beta$ are affine for all $\alpha, \beta \geq \alpha_0$. Then $Y := \varprojlim_{\alpha \in I} Y_\alpha$ exists in the category of schemes.*

Sketch of the proof. For any scheme X, there is a 1:1 correspondence

$$\{\text{quasi-coherent } \mathscr{O}_X\text{-algebras}\} \rightleftarrows \{\text{affine morphisms } X' \to X\}$$
$$\mathscr{S} \mapsto \mathrm{Spec}(\mathscr{S}) \to X$$
$$f_*(\mathscr{O}_{X'}) \leftarrow\!\shortmid f : X' \to X$$

By the above correspondence, we therefore get an inductive system of quasi-coherent $\mathscr{O}_{Y_0}$ algebras $S_\alpha, \alpha \geq \alpha_0$. If we set $\mathscr{S} := \varinjlim_{\alpha \geq \alpha_0} S_\alpha$, then $Y = \mathrm{Spec}(\mathscr{S})$ will be a projective limit of the Y_α. □

In particular, the projective limit of a projective system of affine schemes $(\mathrm{Spec}(A_\alpha))$ always exists and is given as $\mathrm{Spec}(\varinjlim_{\alpha \in I} A_\alpha)$. We will show next that “descending” a scheme X over S to S_{α_0} is equivalent to giving it as the limit of a special kind of projective system. This is often a more convenient point of view.

3.2 Lemma. *Let X be a scheme over S which in turn is given as the projective limit $\varprojlim_{\alpha \in I} S_\alpha$ over the projective system $(S_\alpha, u_{\alpha\beta}, \alpha \geq \beta \in I)$ where all the transition maps are affine. Then the following statements are equivalent:*

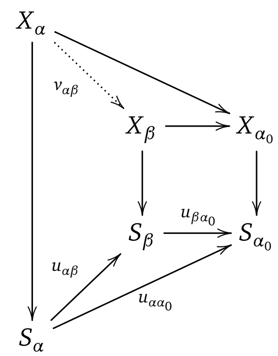
(i) There exists $\alpha_0 \in I$ and a scheme X_{α_0} over S_{α_0} such that $X = X_{\alpha_0} \times_{S_{\alpha_0}} S$.

(ii) There exists $\alpha_0 \in I$ and a projective system of schemes X_α over S_α for $\alpha \geq \alpha_0$ with transition maps $v_{\alpha\beta}$ such that there is a cartesian diagram

$$\begin{array}{ccc} X_\alpha & \xrightarrow{v_{\alpha\beta}} & X_\beta \\ \downarrow & & \downarrow \\ S_\alpha & \xrightarrow{u_{\alpha\beta}} & S_\beta \end{array}$$

for all $\alpha \geq \beta \geq \alpha_0$ and such that $\varprojlim_{\alpha \geq \alpha_0} X_\alpha = X$.

Proof. Let us prove (i)⇒(ii). For $\alpha \geq \alpha_0$, we define a projective system of schemes over S_α for $\alpha \geq \alpha_0$ as follows: we set $X_\alpha := X_{\alpha_0} \times_{S_{\alpha_0}} S_\alpha$. In order to define transition maps $v_{\alpha\beta} : X_\alpha \to X_\beta$ for $\alpha \geq \beta \geq \alpha_0$, we consider the following commutative diagram:



(1)

The map $v_{\alpha\beta}$ exists because the the square

$$\begin{array}{ccc} X_\beta & \longrightarrow & X_{\alpha_0} \\ \downarrow & & \downarrow \\ S_\beta & \xrightarrow{u_{\beta\alpha_0}} & S_{\alpha_0} \end{array} \qquad (2)$$

is cartesian. Furthermore the square

$$\begin{array}{ccc} X_\alpha & \xrightarrow{v_{\alpha\beta}} & X_\beta \\ \downarrow & & \downarrow \\ S_\alpha & \xrightarrow{u_{\alpha\beta}} & S_\beta \end{array}$$

is cartesian, as its composition with the square (2) yields the outer square of (1), which is cartesian by construction. Note that this implies that the transition maps $v_{\alpha\beta}$ of the projective system $X_\alpha, \alpha \geq \alpha_0$ are affine, as this property is stable under base change. Therefore, $\varprojlim_{\alpha \geq \alpha_0} X_\alpha$ exists by Theorem 3.1 and we compute that

$$\varprojlim_{\alpha \geq \alpha_0} X_\alpha = \varprojlim_{\alpha \geq \alpha_0} X_{\alpha_0} \times_{S_{\alpha_0}} S_\alpha = X_{\alpha_0} \times_{S_{\alpha_0}} \left(\varprojlim_{\alpha \geq \alpha_0} S_\alpha \right) = X_{\alpha_0} \times_{S_{\alpha_0}} S = X \ ,$$

where we used that projective limits and fiber products commute (as they do in any category) and the last equality is true by assumption.

In order to prove (ii)⇒(i), note that by assumption, we have

$$X = \varprojlim_{\alpha \geq \alpha_0} X_\alpha = \varprojlim_{\alpha \geq \alpha_0} X_{\alpha_0} \times_{S_{\alpha_0}} S_\alpha = X_{\alpha_0} \times_{S_{\alpha_0}} \left(\varprojlim_{\alpha \geq \alpha_0} S_\alpha \right) = X_{\alpha_0} \times_{S_{\alpha_0}} S$$

where the first equality is true by assumption and the rest follows as in the previous part of the proof. □

3.3 Definition. We call a projective system of schemes as in part (ii) of Lemma 3.2 *cartesian for* $\alpha \geq \alpha_0$.

Lemma 3.2 then says that condition (i) is equivalent to X being the limit of a projective system of scheme over S_α that is cartesian for $\alpha \geq \alpha_0$. This is the setting that is studied in [Gro66]. Before we start stating theorems, let us fix a general set-up.

3.4 Convention. In the following, let X, Y denote finitely presented schemes over an affine scheme S and assume that $S = \lim_{\alpha \in I} S_\alpha$ is the projective limit of affine schemes S_α over a directed poset I with transition maps $u_{\alpha\beta} : S_\alpha \to S_\beta$ for $\alpha \geq \beta$.

3.5 Theorem (cf. [Gro66, Théorème 8.8.2(ii)]). *In the situation of 3.4, assume that X is finitely presented over S. Then for some $\alpha_0 \in I$ there exists a finitely presented scheme X_0 over S_{α_0} such that $X = X_0 \times_{S_{\alpha_0}} Y$.*

As mentioned earlier, theorems like this are helpful, when all the schemes S_α are nicer than S. We will use this particular theorem in the case where $S = \operatorname{Spec}(K)$ for K a field of characteristic 0 and S_α runs over all sub-$\mathbb{Z}$-algebras of K of finite type. In that case, the scheme X_{α_0} is of finite type over $\mathbb{Z}$ and in particular noetherian! By Lemma 3.2 we have then written X as a projective system of noetherian schemes. This is sometimes referred to as *noetherian approximation*. Another possible setting is localization: if $S = \operatorname{Spec}(A_\mathfrak{p})$ with $\mathfrak{p} \in \operatorname{Spec}(A)$, then S_α can run over $\operatorname{Spec}(A_f)$ for all $f \in A \setminus \mathfrak{p}$. We also note that a more general version of the above theorem holds if we replace the system of affine schemes S_α by a general system of schemes that has affine transition maps.

Let us now give a proof of Theorem 3.5 in the case where X is affine. The general case is obtained by a gluing argument (see [Gro66, Théorème 8.8.2(ii)]).

Proof of Theorem 3.5 (affine case). Let us first fix some $\beta_0 \in I$ and replace I by the cofinal subset

$$I' = \{\alpha \in I : \alpha \geq \beta_0\} .$$

The rings $A_\alpha := \Gamma(S_\alpha, \mathscr{O}_{S_\alpha})$ for $\alpha \in I'$ together with the transition maps $u^*_{\alpha\beta} : A_\beta \to A_\alpha$ for $\beta \leq \alpha$ make $(A_\alpha, u^*_{\alpha\beta})$ an inductive system of A_{α_0}-algebras wit limit $\varinjlim_{\alpha \in I'} A_\alpha = \Gamma(S, \mathscr{O}_S) =: A$ and canonical maps $u^*_\alpha : A_\alpha \to A$ for all $\alpha \in I'$. Set $X = \operatorname{Spec}(B)$ for some finitely presented A-algebra B. By assumption, $B = A[x_1, \ldots, x_n]/\mathscr{I}$ where $\mathscr{I}$ is a finitely generated ideal. Fix a set of generators $F_1, \ldots, F_m$ of $\mathscr{I}$. Each F_i is a polynomial with coefficients in A, and for each coefficient c of F_i, we can find some $\alpha_c \in I'$ such that $c = u^*_{\alpha_c}(c')$ for some $c' \in A_{\alpha_c}$. As the set of all coefficients of $F_1, \ldots, F_m$ is finite we can find a α_0 such that all of them arise as image of elements in A_{α_0} under the map $u^*_{\alpha_0}$. Therefore, we can find polynomials $F'_1, \ldots F'_m$ in $A_{\alpha_0}[x_1, \ldots, x_n]$ such that $u^*_{\alpha_0}(F'_i) = F_i$. If we denote by $\mathscr{I}' \subset \mathscr{A}_{\alpha_0}$ the ideal generated by $F'_1, \ldots, F'_m$, it follows that the image of $\mathscr{I}' \otimes_{A_{\alpha_0}} A$ in $A[x_1, \cdots, x_n]$ is exactly $\mathscr{I}$. Now we apply the functor $- \otimes_{A_{\alpha_0}} A$ to the exact sequence

$$\mathscr{I}' \to A_{\alpha_0}[x_1, \ldots, x_n] \to B_{\alpha_0} \to 0$$

to show that $B = B_{\alpha_0} \otimes_{A_{\alpha_0}} A$. The construction also shows that B_{α_0} is finitely presented. □

3.6 Remark. As mentioned before, one deduces Theorem 3.5 from the affine case by a gluing construction. For the argument, it is essential that a morphism of schemes is defined to be of finite presentation if it is locally of finite presentation *and* quasi-compact *and* quasi-separated. Quasi-compactness is used in order to ensure that the α_0 for all the affine pieces have a supremum. Quasi-separatedness ensures that the intersection of the affines is quasi-compact, which makes gluing possible.

Let us now continue by proving another result in the spirit of Theorem 3.5. This time, we want to descend morphisms of schemes. In order to do this, assume we are given schemes X, Y over S and two projective systems of schemes $(X_\alpha, v_{\alpha\beta}), (Y_\alpha, w_{\alpha\beta})$ over the projective system $(S_\alpha, u_{\alpha\beta})$, such that the systems X_α, Y_α are cartesian for $\alpha \geq \alpha_0$ and $\varprojlim_{\alpha\in I} S_\alpha = S, \varprojlim_{\alpha\in I} X_\alpha = X, \varprojlim_{\alpha\in I} Y_\alpha = Y$. By Theorem 3.5 and Lemma 3.2, this happens, for example, if the schemes S_α are affine and X, Y are finitely presented over S. If $\alpha_0 \leq \alpha \leq \beta$, we have maps

$$\begin{aligned} e_{\beta\alpha} : \mathrm{Hom}_{S_\alpha}(X_\alpha, Y_\alpha) &\to \mathrm{Hom}_{S_\beta}(X_\beta, Y_\beta) \\ f &\mapsto f \times_{S_\alpha} \mathrm{id}_{S_\beta} \end{aligned}$$

which make $\mathrm{Hom}_{S_\alpha}(X_\alpha, Y_\alpha), e_{\beta\alpha}, \alpha \geq \alpha_0$ an inductive system of sets. Furthermore, for all $\alpha \geq \alpha_0$ we have maps

$$\begin{aligned} e_\alpha : \mathrm{Hom}_{S_\alpha}(X_\alpha, Y_\alpha) &\to \mathrm{Hom}_S(X, Y) \\ f &\mapsto f \times_{S_\alpha} \mathrm{id}_S \end{aligned}$$

which satisfy $e_\beta \circ e_{\beta\alpha} = e_\alpha$. Therefore, they induce a natural map

$$e : \varinjlim_{\alpha \geq \alpha_0} \mathrm{Hom}_{S_\alpha}(X_\alpha, Y_\alpha) \to \mathrm{Hom}_S(X, Y) \tag{3}$$

3.7 Theorem (cf. [Gro66, Théorème 8.8.2(i)]). *Assume the situation of Convention 3.4, so that the above discussion applies as well. Then the natural map*

$$e : \varinjlim_{\alpha} \mathrm{Hom}_{S_\alpha}(X_\alpha, Y_\alpha) \to \mathrm{Hom}_S(X, Y)$$

from (3) is bijective.

Proof (affine case). Let us show the statement in the case that all schemes $X_\alpha = \mathrm{Spec}(B_\alpha), Y_\alpha = \mathrm{Spec}(C_\alpha)$ are affine. Let $S_\alpha = \mathrm{Spec}(A_\alpha)$. By Theorem 3.5 and Lemma 3.2 we can assume that the projective systems X_α, Y_α are cartesian and that X_α, Y_α are finitely presented over S_α for $\alpha \geq \alpha_0$. The statement we want to prove then looks as follows:

Let A_α be an inductive system of rings such that $\varinjlim_\alpha A_\alpha = A$. Let $B_{\alpha_0}, C_{\alpha_0}$ be A_{α_0}-algebras of finite presentation. Then the canonical map

$$\varinjlim_{\alpha \geq \alpha_0} \mathrm{Hom}_{A_\alpha - alg}(C_{\alpha_0} \otimes_{A_{\alpha_0}} A_\alpha, B_{\alpha_0} \otimes_{A_{\alpha_0}} A_\alpha) \to \mathrm{Hom}_{A - alg}(C_{\alpha_0} \otimes_{A_{\alpha_0}} A, B_{\alpha_0} \otimes_{A_{\alpha_0}} A)$$

is bijective.

First, let us note that we have functorial isomorphisms

$$\mathrm{Hom}_{A_\alpha - \mathrm{alg}}(C_{\alpha_0} \otimes_{A_{\alpha_0}} A_\alpha, B_{\alpha_0} \otimes_{A_{\alpha_0}} A_\alpha) \cong \mathrm{Hom}_{A_{\alpha_0} - \mathrm{alg}}(C_{\alpha_0}, B_{\alpha_0} \otimes_{A_{\alpha_0}} A_\alpha) \tag{4}$$

$$\mathrm{Hom}_{A - \mathrm{alg}}(C_{\alpha_0} \otimes_{A_{\alpha_0}} A, B_{\alpha_0} \otimes_{A_{\alpha_0}} A) \cong \mathrm{Hom}_{A_{\alpha_0} - \mathrm{alg}}(C_{\alpha_0}, B_{\alpha_0} \otimes_{A_{\alpha_0}} A) \tag{5}$$

which are constructed as follows: we have maps

$$\begin{aligned} \mathrm{Hom}_{A_\alpha - \mathrm{alg}}(C_{\alpha_0} \otimes_{A_{\alpha_0}} A_\alpha, B_{\alpha_0} \otimes_{A_{\alpha_0}} A_\alpha) &\overset{s}{\underset{t}{\rightleftarrows}} \mathrm{Hom}_{A_{\alpha_0} - \mathrm{alg}}(C_{\alpha_0}, B_{\alpha_0} \otimes_{A_{\alpha_0}} A_\alpha) \\ f &\mapsto s(f);\ s(f)(x) := f(x \otimes 1) \\ t(g)(x \otimes y) := g(x) \cdot y\ ; t(g) &\leftarrow g \end{aligned} \tag{4}$$

which are mutually inverse (similarly for (5)). We are therefore reduced to proving the following:

Let G, F be E-algebras such that $F = \varinjlim_{\alpha\ ini} F_\alpha$. Then the natural map

$$\varinjlim_{\alpha \in I} \mathrm{Hom}_E(G, F_\alpha) \to \mathrm{Hom}_E(G, F)$$

which assigns to each inductive system of maps f_α its inductive limit is bijective.

We first show injectivity: since G is a finitely presented algebra by assumption, it will in particular be finitely generated. Let $(t_i)_{0 \le i \le n}$ be a system of generators for G and fix two inductive systems of maps $(f_\alpha), (g_\alpha)$ such that $f := \varinjlim f_\alpha = \varinjlim g_\alpha =: g$. Let us show that there exists some μ such that $f_\mu = g_\mu$ (which is equivalent to $(f_\alpha) = (g_\alpha)$ in $\varinjlim_{\alpha \ge \alpha_0} \mathrm{Hom}_{E-\mathrm{alg}}(G, F_\alpha)$). Let us denote by $\varphi_{\beta\alpha} : F_\alpha \to F_\beta$ for $\beta \ge \alpha$ and $\varphi_\alpha : F_\alpha \to F$ the canonical maps. By hypothesis, for each i, there exists λ_i such that $\varphi_{\lambda_i}(f_{\lambda_i}(t_i)) = \varphi_{\lambda_i}(g_{\lambda_i}(t_i))$, and by taking $\lambda \ge \lambda_i$ for all i, we actually have $\varphi_\lambda(f_\lambda(t_i)) = \varphi_\lambda(g_\lambda(t_i))$ for $0 \le i \le n$. Thus, there exists $\mu \ge \lambda$ such that $\varphi_{\mu\lambda}(f_\lambda(t_i)) = \varphi_{\mu\lambda}(g_\lambda(t_i))$ for $0 \le i \le n$. Therefore, we have that $f_\mu(t_i) = g_\mu(t_i)$ for $0 \le i \le n$, from which it follows that $f_\mu = g_\mu$. (Note that this step only needed that G is finitely generated.)

In order to prove surjectivity, we will need the full assumption that G is finitely presented: write

$$G = E[T_1, \ldots, T_n]/\mathscr{J}$$

where $\mathscr{J}$ is a finitely generated ideal such that $t_i \equiv T_i \mod \mathscr{J}$. Let $(P_j)_{1 \le j \le m}$ be a system of generators for $\mathscr{J}$. If we're given a morphism of E-algebras $\theta : G \to F$, we want to show that $\theta = \varinjlim_\alpha \theta_\alpha$ for some inductive system of maps $(\theta_\alpha)_{\alpha \ge \alpha_0}$. Let $b_i := \theta(t_i)$, then by definition we have $P_j(b_1, \ldots, b_n) = \theta(P_j(t_1, \ldots, t_n)) = 0$. Furthermore, there exists a β and $x_1, \ldots, x_n \in F_\beta$ such that $\varphi_\beta(x_i) = b_i$ for $1 \le i \le n$ and therefore $\varphi_\beta(P_j(x_1, \ldots, x_n)) = P_j(b_1, \ldots, b_n) = 0$ for $1 \le j \le m$. It follows that there exists a $\gamma \ge \beta$ such that $\varphi_{\gamma\beta}(P_j(x_1, \ldots, x_n)) = P_j(\varphi_{\gamma\beta}(x_1), \ldots, \varphi_{\gamma\beta}(x_n)) = 0$ for $1 \le j \le m$. We conclude that there exists a morphism of E-algebras $\theta_\gamma : G \to F_\gamma$ defined by $\theta(\gamma)(t_i) = \varphi_{\gamma\beta}(x_i)$. For $\delta \ge \gamma$, we define $\theta_\delta : G \to F_\delta$ as the composition $\varphi_{\delta\gamma} \circ \theta_\gamma$. We have then defined an inductive system of maps $\theta_\delta, \delta \ge \gamma$ such that $\varinjlim_{\delta \ge \gamma} \theta_\delta = \theta$. □

3.8 Corollary. *Assume the situation of 3.4 and let $f : X \to Y$ be an S-morphism. Then there exists $\alpha_0 \in I$ and finitely presented schemes $X_{\alpha_0}, Y_{\alpha_0}$ over S_{α_0} and a S_{α_0}-morphism $f_{\alpha_0} : X_{\alpha_0} \to Y_{\alpha_0}$ such that $f = f_{\alpha_0} \times_{S_{\alpha_0}} \mathrm{id}_S$.*

Proof. By Theorem 3.7, f is the inductive limit of an inductive system of morphisms $(f_\alpha : X_\alpha \to Y_\alpha)_{\alpha \in I}$. By the discussion preceding Theorem 3.7, the maps inducing e as in 3 are given for $\beta \ge \alpha_0$ as $e_\beta(f_{\alpha_0}) = f_\beta \times_{S_\beta} \mathrm{id}_S$. It follows that $f = e((f_\alpha)_{\alpha \in I}) = e_{\alpha_0}(f_{\alpha_0}) = f_{\alpha_0} \times_{S_{\alpha_0}} \mathrm{id}_S$. □

In view of Corollary 3.8 it is natural to ask whether f_{α_0} can be chosen in such a way that it shares some desirable properties of f, i.e. whether we can “descend” properties of f to f_{α_0}. This ties in with the philosophy that properties of schemes are really properties of their structural morphisms. The answer to the question is positive for many useful properties of f. Note that one would naturally expect that those properties are stable under base change.

3.9 Theorem. *Let X be an S-scheme of finite presentation. If X is smooth (resp. projective, proper) over S, then there exists an α_0 and a smooth (resp. projective, proper) scheme X_{α_0} of finite presentation over S_{α_0} such that $X = X_{\alpha_0} \times_{S_{\alpha_0}} S$.*

The proofs of these statements are of varying degrees of difficulty. If X is of finite presentation and projective over $S = \mathrm{Spec}(A)$, this means that X is a closed subscheme of $\mathbb{P}^n_A$ which is cut our by a finitely generated homogeneous ideal of $A[X_1, \ldots, X_s]$. But all the generators of this ideal are images of generators of a homogeneous ideal in $A_{\alpha_0}[X_1, \ldots, X_s]$ that determines a closed subscheme of $\mathbb{P}^n_{A_{\alpha_0}}$. This is the projective scheme over X_{α_0} we are looking for. The proofs for smooth and proper schemes are more complicated and we will skip them.

Let us finish the discussion by stating that we can also descend finitely presented $\mathcal{O}_X$-modules over finitely presented S-schemes, as well as morphisms between them.

3.10 Theorem. *Assume the situation of Convention 3.4.*

(i) If E is a finitely presented $\mathcal{O}_X$-module, then there exists a finitely presented $\mathcal{O}_{X_\alpha}$-module E_{α_0} such that E is induced from E_{α_0} by base change with S. Furthermore, the following properties can be descended from E to E_{α_0}: locally free (of rank r), (very) ample invertible (if X is projective and we choose X_{α_0} to be projective as well by Theorem 3.9).

(ii) Let E, F be finitely presented $\mathcal{O}_X$-modules. Then there exists a cartesian inductive system of finitely presented S_α-schemes X_α and cartesian systems of $\mathcal{O})_{X_\alpha}$-modules E_α, F_α such that the natural map

$$\varinjlim_{\alpha} \mathrm{Hom}_{\mathcal{O}_{X_\alpha}}(E_\alpha, F_\alpha) \to \mathrm{Hom}_{\mathcal{O}_X}(E, F)$$

is bijective.

4 Finiteness of residue fields, density of closed points and smooth locus

The techniques discussed up to this point will be combined with the following statements.

4.1 Theorem (see [Gro66, 10.4])**.** *Let S be a scheme of finite type over $\mathbb{Z}$.*

(i) If x is a closed point of S, the residue field $\mathrm{k}(x)$ *is a finite field.*

(ii) All locally closed non-empty subsets $Z \subset S$ contain a closed point of S.

Sketch of the proof. By assumption, $S = \bigcup_{i=1}^n \mathrm{Spec}(A_i)$, where A_i is a finitely generated $\mathbb{Z}$-algebra, i.e. there is a surjection $\mathbb{Z}[x_1, \ldots x_{n_i}] \twoheadrightarrow A_i$. The first statement then follows from a standard result in commutative algebra ("finitely generated fields are finite fields").

By definition, a scheme whose underlying topological space X satisfies condition (ii) is a *Jacobson scheme*. A scheme is Jacobson if and only if it can be covered by opens $\mathrm{Spec}(A_i)$ such that all A_i are Jacobson rings (i.e. every prime ideal in A_i is an intersection of maximal ideals). Finitely generated $\mathbb{Z}$-algebras are of this type. □

Theorem 4.1 can easily fail if S is not of finite type over $\mathbb{Z}$.

4.2 Example. Let K be an infinite field. The residue field at the closed point of $\mathrm{Spec}(K)$ is the field K which is not finite.

4.3 Example. Let R be any local ring of Krull dimension ≥ 1. Then $\operatorname{Spec}(R)$ has a unique closed point $\mathfrak{m}$. The set $U = \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ is nonempty and open, and therefore certainly locally closed in $\operatorname{Spec}(R)$. However, U does not contain a closed point of $\operatorname{Spec}(R)$. Also note that there exist schemes that do not have closed points at all (see [Sch05])!

Let us finish by stating a theorem which tells us that an integral scheme of finite type over $\operatorname{Spec}(\mathbb{Z})$ is generically smooth.

4.4 Proposition. *Let S be an integral scheme of finite type over* $\operatorname{Spec}(\mathbb{Z})$*. The set of points where S is smooth over* $\operatorname{Spec}(\mathbb{Z})$ *is a non-empty open (i.e. dense) set. In particular, if A is a finite type $\mathbb{Z}$-algebra and integral, there exists $s \in A$ such that* $\operatorname{Spec}(A_s)$ *is smooth over* $\operatorname{Spec}(\mathbb{Z})$*.*

4.5 Remark. In the proof of H2dR in characteristic 0, the techniques from Section 3 will allow us to pass from a smooth and proper scheme over a field K of characteristic 0 to a smooth and proper scheme over $\operatorname{Spec}(A)$ where A is a finite type $\mathbb{Z}$-algebra. Theorem 4.4 will then make it possible to replace A by an algebra A_s which is smooth over $\operatorname{Spec}(\mathbb{Z})$, and by Theorem 4.1, $\operatorname{Spec}(A_s)$ contains a closed point of $\operatorname{Spec}(A)$ which has finite residue field. This will allow us to conclude with a characteristic p argument.

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