Hodge-to-de Rham degeneration
Lecture 2: Additional properties of smoothness

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March 13, 2015

1 Preliminaries

In order to prove some of the properties of smooth, unramified and étale morphisms, we need to revise some results on the sheaf of relative differentials. The main reference for this section is [3, Section 1].

Definition 1.1. Let $i : X \to Z$ be an immersion of associated ideal $\mathcal{I}$. Consider the subscheme Z_1 of Z with the same underlying space as X and defined by the ideal $\mathcal{I}^2$. Then one has that j factors in a unique way into

$$X \xrightarrow{i_1} Z_1 \xrightarrow{h_1} Z$$

where h_1 is an immersion and i_1 is a thickening of order 1 with associated ideal $\mathcal{I}/\mathcal{I}^2$.

The pair (i_1, h_1), or just Z_1, is called the *first infinitesimal neighbourhood* of i. The ideal $\mathcal{I}/\mathcal{I}^2$ is called the *conormal sheaf* of i and is usually denoted by $\mathcal{N}_{X/Y}$.

Consider a morphism of schemes $f : X \to Y$ and the associated diagonal morphism $\Delta : X \to X \times_Y X$. Observe that, by definition, the sheaf of relative differentials $\Omega^1_{X/Y}$ is conormal sheaf of Δ, as $\Omega^1_{X/Y} = \mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal associated to the immersion Δ.

Consider now $X \xrightarrow{\Delta_1} Z_1 \xrightarrow{h_1} X \times_Y X$ the first infinitesimal neighbourhood of Δ.

Definition 1.2. The structure sheaf of the first infinitesimal neighbourhood of the diagonal is called the sheaf of *principal parts of order 1* of X over Y, and it is usually denoted by $\mathcal{P}^1_{X/Y}$.

Remark 1.3. Observe that by construction we have the following exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \to \Omega^1_{X/Y} \to \mathcal{P}^1_{X/Y} \to \mathcal{O}_X \to 0 \tag{1}$$

Note that if we compose the two projections of $X \times_Y X$ on X with $h_1 : Z_1 \to X \times_Y X$ we obtain two morphisms $p_1, p_2 : Z_1 \to X$ that provide a retraction to the thickening Δ_1. Denote by $j_1, j_2 : \mathcal{O}_X \to \mathcal{P}^1_{X/Y}$ the associated morphisms of sheaves to p_1

and p_2. They provide and splitting of (1). Hence (1) is an split exact sequence of $\mathcal{O}_X$-modules (and in particular of $f^{-1}(\mathcal{O}_Y)$-modules).

This also provides a way of defining globally the differential $d_{X/Y}$.

Definition 1.4. We define $d_{X/Y} : \mathcal{O}_X \to \Omega^1_{X/Y}$ to be the difference $j_2 - j_1$ which takes values on Ω^1_X by the definition of j_1 and j_2.

Proposition 1.5. *Given a morphism of schemes $f : X \to Y$, the infinitesimal neighbourhood of order 1 of $\Delta : X \to X \times_Y X$ parametrizes the pairs of Y-points of X congruent modulo an ideal of square zero. More precisely, if $i : T_0 \to T$ is a thickening of order 1 with associated ideal $\mathcal{I}$ where T is a Y-scheme, and $t_1, t_2 : T \to X$ are two morphisms of Y-schemes that coincide modulo $\mathcal{I}$, i.e. $t_1 \circ i = t_2 \circ i = t_0 : T_0 \to X$, then there exists a unique Y-morphism $h : T \to Z_1$ such that $p_1 \circ h = t_1$ and $p_2 \circ h = t_2$.*

In addition, if $t_1^*, t_2^* : \mathcal{O}_X \to t_{0*}\mathcal{O}_T$ are the homomorphisms of sheaves associated to t_1 and t_2, $t_2^* - t_1^* \in \mathrm{Der}_Y(\mathcal{O}_X, t_{0*}\mathcal{I})$ such that $(t_2^* - t_1^*)(s) = h^*(ds) = h^*((j_2 - j_1)(s))$.

2 Basic properties of smooth, unramified and étale morphisms

2.1 Extensions

First we recall the definition of extensions of modules over a ring and their properties. The main reference we have followed for the first part of the present section is [5, Section 3.4].

Definition 2.1. Let R be a ring and A, B two R-modules. An R-extension of A by B is an exact sequence

$$\xi : 0 \to B \to E \to A \to 0$$

of R-modules.

Definition 2.2. We say that two R-extensions ξ, ξ' of A by B are *equivalent* if there is a commutative diagram

$$\begin{tikzcd}
\xi: & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\varphi", "\cong"'] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi': & 0 \arrow[r] & B \arrow[r] & E' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

of R-modules.

We say an extension is *split* if it is equivalent to $0 \to B \to A \oplus B \to A \to 0$.

Definition 2.3. Given $\xi : 0 \to B \xrightarrow{i} E \xrightarrow{p} A \to 0$ and $\xi' : 0 \to B \xrightarrow{i'} E' \xrightarrow{p'} A \to 0$ two R-extensions of A by B, we can define the *Baer sum* of ξ and ξ' as follows. Take $E'' = E \times_A E' = \{(x, x') \in E \times E', p(x) = p(x')\}$ the pullback in the category $\mathrm{Mod}(R)$. Observe that the elements of the skew diagonal $D = \{(i(b), -i'(b))\} \in E \times E'$ form an R-submodule of E'' and that they are sent to zero in A. Let $F = E''/D$, then one can easily check that

$$0 \to B \to F \to A \to 0$$

is an R-extension of A by B. We call it the Baer sum of ξ and ξ'.

The zero element for Baer sum is the equivalence class of split extensions of A by B.

It can be checked that the definition of Baer sum behaves well with respect to the equivalence relation defined above, hence we get the following:

Proposition 2.4. *The set of equivalent classes of R-extensions of A by B can be endowed with the structure of an abelian group by means of the* Baer sum. *The zero element under the Baer sum is the equivalence class of split extensions.*

Theorem 2.5. *Let A, B be two modules over a ring R. Then there is a group isomorphism:*

$$\{\textit{equivalent classes of } R\textit{-extensions of } A \textit{ by } B\} \to \mathsf{Ext}^1_R(A,B)$$

Remark 2.6. The definitions above can be given analogously for quasi-coherent sheaves over a scheme. Thus, given a scheme X and two quasi-coherent rings $\mathcal{F}, \mathcal{G}$ of $\mathcal{O}_X$-modules, we can talk of $\mathcal{O}_X$-extensions of $\mathcal{F}$ by $\mathcal{G}$, their equivalence classes, the Baer sum and we also have the group isomorphism:

$$\{\text{equivalent classes of } \mathcal{O}_X\text{-extensions of } \mathcal{F} \text{ by } \mathcal{G}\} \to \mathsf{Ext}^1_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$$

defined analogously as in the case for modules. Note that we are not using sheaf cohomology, otherwise it is in general not true that local extensions glue to a global extension in the sheaf cohomology.

Now we provide a geometric definition of another type of extensions and its properties. For this part of the notes we have strongly followed [3] and [2].

Definition 2.7. Let $f : X \to Y$ a morphism of schemes and let $\mathcal{I}$ be a quasi-coherent $\mathcal{O}_X$-module. A Y-extension of X by $\mathcal{I}$ is a morphism $i : X \to X'$ over Y which is a thickening of order 1 with ideal $\mathcal{I}$.

Definition 2.8. Given two Y-extensions $i_1 : X \to X_1$ and $i_2 : X \to X_2$ of X by $\mathcal{I}$, we say that they are equivalent if there exists an Y-isomorphism $g : X_1 \to X_2$ such that the diagram

$$\begin{tikzcd} X \arrow[d, "i_1"] \arrow[dr, "i_2"] & \\ X_1 \arrow[r, "g"] & X_2 \end{tikzcd}$$

is commutative and it induces the identity on $\mathcal{I}$.

We will denote by $\mathsf{Ext}_Y(X,\mathcal{I})$ the set of equivalent classes of Y-extensions of X by $\mathcal{I}$.

Remark 2.9. This notion coincides in the affine setting with the notion of a *(square zero) extension of an algebra* over a ring by a bimodule [5, p. 311].

As it happened with module extensions, we can also construct a "Baer sum" in the set $\mathsf{Ext}_Y(X,\mathcal{I})$ such that it becomes an abelian group where the zero element is the equivalence class of the trivial Y-extension of X by $\mathcal{I}$, i.e. it is the canonical morphism $i : X \to X'$, where X' is the scheme with the same underlying topological space as X and with structure sheaf $\mathcal{O}_X \oplus \mathcal{I}$ the idealization of $\mathcal{I}$.

2.2 Properties of smooth, unramified and étale morphisms

Proposition 2.10. *The following properties hold:*

1. *Let $f : X \to Y$ be a morphism of schemes. If f is smooth (resp. unramified) then $\mathcal{O}_X$-module $\Omega^1_{X/Y}$ is locally free of finite type (resp. zero).*

2. *Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes. If f is smooth, then the* first fundamental exact sequence *extended by a zero to the left:*

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0 \tag{2}$$

   is exact and locally split.
   In particular if f is étale $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is an isomorphism.

3. *Consider the commutative diagram of morphisms of schemes*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Z \arrow[dl, "g"] \\ Y & \end{tikzcd} \tag{3}$$

   where i is a closed immersion with associated ideal $\mathcal{I}$. If f is smooth, then the second fundamental exact sequence *extended by a zero to the left:*

$$0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{Z/Y} \to \Omega^1_{X/Y} \to 0 \tag{4}$$

   is exact and locally split.
   In particular, if f is étale the canonical homomorphism $\mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{Z/Y}$ is an isomorphism.

We will provide a proof following as main reference [3].

First we prove 3. for f smooth:

Assume f is smooth and denote $X \xrightarrow{i_1} Z_1 \xrightarrow{h_1} Z$ the first infinitesimal neighbourhood of $i : X \to Z$. Consider the diagram

$$\begin{tikzcd} & & X \arrow[r, "i"] \arrow[d, "f"] & Z \arrow[dl, "g"] \\ X \arrow[urr, "\mathrm{Id}"] \arrow[r, "i_1"'] & Z_1 \arrow[r, "h_1"'] \arrow[ur, dotted, "r"] & Y & \end{tikzcd}$$

where we know, by definition of smooth morphism, that such an $r : Z_1 \to X$ exists Zarisky-locally in Z_1 making the diagram commute. Then we have that morphisms $t_1 = h_1 \circ i_1 \circ r$, $t_2 = h_1 : Z_1 \to Z$ locally defined in Z_1 coincide locally when restricted to X, as we have $t_1 \circ i_1 = i = t_2 \circ i_1$ locally. By means of Proposition 1.5, we have that $t_2^* - t_1^*$ is locally an Y-derivation of $\mathcal{O}_Z$ in $i_*(\mathcal{I}/\mathcal{I}^2)$). Hence we have that $t_2^* - t_1^*$ induces locally a morphism $s : i^*\Omega^1_{Z/Y} \to \mathcal{I}/\mathcal{I}^2$ of $\mathcal{O}_X$-modules via

$$\mathrm{Der}_Y(\mathcal{O}_Z, i_*(\mathcal{I}/\mathcal{I}^2)) \cong \mathrm{Hom}_{\mathcal{O}_Z}(\Omega^1_{Z/Y}, i_*(\mathcal{I}/\mathcal{I}^2)) \cong \mathrm{Hom}_{\mathcal{O}_X}(i^*\Omega^1_{Z/Y}, \mathcal{I}/\mathcal{I}^2)$$

and one can check that s gives us a local retraction of the canonical $\mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{Z/Y}$, which gives us the locally split exactness of the short exact sequence.

Now we will proof two lemmas that will provide us the key ingredients for the rest of the proof.

Lemma 2.11. *Let $f : X \to Y$ be a smooth morphism of schemes. Consider the morphism*

$$e : \mathsf{Ext}_Y(X, \mathcal{I}) \to \mathsf{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{I})$$

that associates to every Y-extension $i : X \to Z$ of X by a quasi-coherent $\mathcal{O}_X$-module $\mathcal{I}$, the exact sequence of $\mathcal{O}_X$-modules

$$e(i) : 0 \to \mathcal{I} \to i^*\Omega^1_{Z/Y} \to \Omega^1_{X/Y} \to 0$$

Then, the morphism e is an isomorphism of groups.

Proof. We will give a sketch of the proof as it appears in [3]. First of all, by Proposition 2.10.3. we have that the sequence (4) is split exact. But as $i : X \to Z$ is an Y-extension of X by $\mathcal{I}$, by definition we have that $\mathcal{I}^2 = 0$ in Z, which gives us the exact sequence of $\mathcal{O}_X$-modules $e(i)$ above. Then, one can check that the morphism e preserves “Baer sum”.

It remains to prove that it is an isomorphism. An inverse can be constructed as follows. Take

$$0 \to \mathcal{I} \to \mathcal{M} \xrightarrow{p} \Omega^1_{X/Y} \to 0$$

an element in $\mathsf{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{I})$. Consider now the canonical split exact sequence of $\mathcal{O}_X$-modules

$$0 \to \Omega^1_{X/Y} \to \mathcal{P}^1_{X/Y} \to \mathcal{O}_X \to 0$$

where $\mathcal{P}^1_{X/Y}$ is the sheaf of principal parts of order 1 of X over Y. Via the morphism $j_1 : \mathcal{O}_X \to \mathcal{P}^1_{X/Y}$ we have an isomorphism $\mathcal{O}_X \oplus \Omega^1_{X/Y} \xrightarrow{\cong} \mathcal{P}^1_{X/Y}$ of $\mathcal{O}_X$-modules and in particular of $f^{-1}(\mathcal{O}_Y)$-modules. Then, if we take $\mathcal{F} := \mathcal{O}_X \oplus \mathcal{M}$, we get a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{F} \xrightarrow{\mathrm{Id}\oplus p} \mathcal{P}^1_{X/Y} \to 0$$

which provides an element in $\mathsf{Ext}^1_{f^{-1}(\mathcal{O}_Y)}(\mathcal{P}^1_{X/Y}, \mathcal{I})$.

Notice that if we consider $\mathcal{F}$ as the trivial extension (or the so called idealisation of $\mathcal{M}$) with the structure of $f^{-1}(\mathcal{O}_Y)$-algebra given by the product $(a, m)\cdot(a', m') = (a\cdot a', a\cdot m' + a'\cdot m)$, we have that the morphism of schemes $X \to X'$ (where X and X' have the same underlying topological space) given by $\mathrm{Id}\oplus p$ is a thickening of ideal $\mathcal{I}$. If then one considers the image of this extension by $j_2 = d_{X/Y} + j_1 : \mathcal{O}_X \to \mathcal{P}^1_{X/Y}$ (see [2, Section 1.1.3.]) one gets an $f^{-1}(\mathcal{O}_Y)$- extension of $\mathcal{O}_X$ by $\mathcal{I}$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{O}_X \arrow[r] \arrow[d, "j_2"] & 0 \\
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \oplus \mathcal{M} \arrow[r, "\mathrm{Id}\oplus p"] & \mathcal{P}^1_{X/Y} \arrow[r] & 0
\end{tikzcd}$$

This extension defines a thickening of order one of $\mathcal{O}_X$ by the ideal $\mathcal{I}$. The morphism of schemes $i : X \to Z$ associated to $\mathcal{E} \to \mathcal{O}_X$ (X and Z have the same underlying topological space) is an element of $\mathsf{Ext}_Y(X, \mathcal{I})$. This construction gives us the inverse of the morphism e and it can also be checked to preserve the "Baer sum", which completes the proof. $\square$

Lemma 2.12. *Let $f : X \to Y$ be a morphism locally of finite presentation between affine schemes (i.e. it corresponds to a morphism of rings $A \to B$ making B a an A-algebra of finite presentation). Then f is smooth if and only if $\mathrm{Ext}_Y(X, \mathcal{I}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{I}$.*

Proof. Assume $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$ with A, B rings and B finitely presented A-algebra. The condition of the morphism f to be smooth in this affine setting can be translated to the following. The morphism $f : X \to Y$ is smooth if for any commutative diagram

$$\begin{tikzcd}
 & & B \arrow[dll] \arrow[dl, dotted, "r"] \\
C/I & C \arrow[l] & A \arrow[l] \arrow[u, "f"']
\end{tikzcd}$$

with C an A-algebra and I an A-submodule of C such that $I^2 = 0$ in C, then there exists an arrow $r : B \to C$ of A-algebras that makes the diagram commutative.

Giving $i : X \to X_0$ an Y-extension of X by $\mathcal{I}$ in the affine case is reduced to giving an A-algebra C and a surjection $p : C \to C/I$ where I is an A-submodule of C such that $I^2 = 0$ inside C and $B \cong C/I$ as an A-algebra. The trivial Y-extension of X by $\mathcal{I}$ is in the affine case given by $B \oplus I$ with the trivial extension structure of A-algebra.

Assume f is smooth, then for any Y-extension $p : C \to C/I \cong B$ of X by $\mathcal{I} = \widetilde{I}$ (sheaf of modules associated to I) we have the following diagram

$$\begin{tikzcd}
 & & B \arrow[dll, "\cong"'] \arrow[dl, dotted, "r"] \\
C/I & C \arrow[l] & A \arrow[l] \arrow[u, "f"']
\end{tikzcd}$$

Then one can easily check that the morphism $r \oplus i : B \oplus I \to C$ is an isomorphism of A-algebras where $i : I \hookrightarrow C$ is the natural injection. It reduces to use the fact that r is a retraction of $C \to B$. Hence one deduces immediately that the extension C is equivalent to the trivial extension, which concludes one direction of the argument.

Now assume $\mathrm{Ext}_Y(X, \mathcal{I})$ is locally trivial. Consider any thickening of order 1 over Y, i.e. an A-algebra C and a A-submodule I of C such that $I^2 = 0$ in C, and assume we have the following commutative diagram:

$$\begin{tikzcd}
 & & B \arrow[dll, "g"'] \\
C/I & C \arrow[l] & A \arrow[l] \arrow[u, "f"']
\end{tikzcd}$$

Take the exact sequence

$$0 \to I \to C \to C/I \to 0$$

Observe that as C/I is a B-algebra and $I^2 = 0$, we get an structure of B-module on I. Applying the image of this extension of algebras by the morphism $g : B \to C/I$

[2, Section 1.1.3.], we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & B \times_{C/I} C \arrow[r, "p'"] \arrow[d, "q"] & B \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & I \arrow[r] & C \arrow[r, "p"] & C/I \arrow[r] & 0
\end{tikzcd}$$

where the upper row provides a Y-extension of X by $\mathcal{I} = \widetilde{I}$. By hypothesis, this extension is trivial, hence the exact sequence is split exact and we can take a retraction $h : B \to B \times_{C/I} C$ of p'. Then it follows from the commutativity of the diagram and the fact that $p \circ h = \mathrm{Id}_B$ that the morphism $r = p \circ q \circ h : B \to C/I$ makes the following diagram commutative

$$\begin{tikzcd}
 & & B \arrow[lld, "\cong"'] \arrow[ld, "r"] \\
C/I & C \arrow[l] & A \arrow[l] \arrow[u, "f"']
\end{tikzcd}$$

Thus $f : X = \mathsf{Spec}(B) \to Y = \mathsf{Spec}(A)$ is smooth. $\square$

We can now procceed to prove the rest of the theorem.

First we prove point 1:

Assume $f : X \to Y$ is smooth. As smoothness is a local notion in the source and the target, by Lemma 2.12 we have that every Y-extension of X by any quasi-coherent $\mathcal{O}_X$-module $\mathcal{I}$ is locally trivial. Hence we have that $\mathsf{Ext}^1_{\mathcal{O}_U}(\Omega^1_{U/Y}, \mathcal{I}|_U) = 0$ for all open subsets U of X and all $\mathcal{I}$ quasi-coherent $\mathcal{O}_X$-coherent modules as a direct consequence of Lemma 2.11. Thus the sheaf $\underline{\mathsf{Ext}}^1_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{I})$ associated to the presheaf $U \mapsto \mathsf{Ext}^1_{\mathcal{O}_U}(\Omega^1_{U/Y}, \mathcal{I}|_U)$ is zero for all quasi-coherent $\mathcal{O}_X$-module $\mathcal{I}$ and consequently we have that $\underline{\mathsf{Ext}}^1_{\mathcal{O}_U}(\Omega^1_{U/Y}, \mathcal{J}) = 0$ for all U open subset of X and all $\mathcal{O}_U$-module $\mathcal{J}$. This fact and the fact that $\Omega^1_{X/Y}$ is of finite type, implies that $\Omega^1_{X/Y}$ is a locally free $\mathcal{O}_X$-module of finite type. Indeed, we can find a covering of X by open affine subsets U_i such that $\mathsf{Ext}^1_{\mathcal{O}_{U_i}}(\Omega^1_{U_i/Y}, \mathcal{J}_i) = \underline{\mathsf{Ext}}^1_{\mathcal{O}_{U_i}}(\Omega^1_{U_i/Y}, \mathcal{J}_i) = 0$ for all quasi-coherent $\mathcal{O}_{U_i}$-modules $\mathcal{J}_i$. Observe that we have, for each U_i, the following exact sequence

$$0 \to \mathrm{Ker}(p) \to \mathcal{O}^r_{U_i} \xrightarrow{p} \Omega^1_{U_i/Y} \to 0$$

Fruthermore we have that $\mathsf{Ext}^1_{\mathcal{O}_{U_i}}(\Omega^1_{U_i/Y}, \mathcal{J}_i) = 0$ for all quasi-coherent $\mathcal{O}_{U_i}$-modules $\mathcal{J}_i$ if and only if this short exact sequence is split for all U_i. Hence, for all U_i in the covering, we have that $\Omega^1_{U_i/Y}$ is a direct summand of a free $\mathcal{O}_{U_i}$-module and thus a projective $\mathcal{O}_{U_i}$-module. As projectives in an affine scheme are locally free, we deduce that $\Omega^1_{U_i/Y}$ is locally free for all U_i and thus so it is $\Omega^1_{X/Y}$.

Assume now $f : X \to Y$ is unramified, which implies, by definition, that at most there is one retraction of the trivial extension. As a consequence of Proposition 1.5, one can prove that for any Y-scheme and given the trivial Y-extension $i : X \to Z$ of X by a quasi-coherent $\mathcal{O}_X$-module $\mathcal{I}$ we have a bijection

$$\{Y\text{-retractions of } Z\} \to \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{I}) : r \mapsto r - r_0$$

where r_0 is the retraction associated to the natural injection $\mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{I}$. As the retraction provided by r_0 always exists, we have that it is the only one, hence for all quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{I}$, $\mathrm{Hom}_{\mathcal{O}_{X'}}(\Omega^1_{X'/Y}, \mathcal{I}) = 0$ holds, from what we conclude that $\Omega^1_{X/Y} = 0$.

Now we prove the statement for f étale in point 3:

If f is étale, in particular it is smooth and unramified. By point 3, we already know that the sequence (4) is exact and locally split. By point 1, we know that $\Omega^1_{X/Y} = 0$, hence we have that the canonical morphism $\mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{Z/Y}$ is an isomorphism.

We finish the argument by proving point 2:

Again, as the property of being smooth is local in the source and the target, we can reduce to proving the statement for the case where X, Y and S are affine. Then, we have that

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

is exact and split if and only if

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_X}(f^*\Omega^1_{X/Y}, \mathcal{I}) \to 0$$

is exact for all quasi-coherent $\mathcal{O}_X$-module $\mathcal{I}$. We know that this holds if and only if $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{I}) = 0$ for all quasi-coherent $\mathcal{O}_X$-module $\mathcal{I}$. Thus, if f is smooth, we conclude by Lemma 2.11 and Lemma 2.12.

Assume now f is étale, then in particular it is smooth hence the sequence (3) is exact and locally split. In addition by point 1, we have that $\Omega^1_{X/Y} = 0$, hence we can conclude that $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is an isomorphism. This finishes the proof of Proposition 2.10.

There are converse statements for Proposition 2.10.2. and Proposition 2.10.3. which provide a nice criterion of smoothness and étaleness for morphisms of sheaves. We state them here, but we will not provide a proof.

Proposition 2.13. *The following properties hold:*

1. *Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes and assume $g \circ f$ is smooth. If the sequence*
$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$
is exact and locally split, then f is smooth.

   In particular, if the canonical homomorphism $f^*\Omega^1_{X/Y} \to \Omega^1_{X/S}$ is an isomorphism, then f is étale.

2. *Consider the commutative diagram of morphisms of schemes*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Z \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

   where i is a closed immersion with associated ideal $\mathcal{I}$ and assume g is smooth. If the sequence
$$0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{Z/Y} \to \Omega^1_{X/Y} \to 0$$

is exact and locally split, then f is smooth.

In particular, if the canonical homomorphism $\mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{Z/Y}$ is an isomorphism, then f is étale.

3 Local and global structure of smooth morphisms

The properties of smooth morphisms explained in the previous section allow us to study the structure of smooth morphisms locally. The main references for this section are [3] and [1].

Proposition 3.1. *Let $f : X \to Y$ be a smooth morphism. Let $x \in X$ be a point of X. Then there exists an open neighbourhood U of x and an étale morphism $s : U \to \mathbb{A}^n_S$ for some integer $n \geq 0$ such that $f|_U$ is the composition $U \to \mathbb{A}^n_s \to S$, where the second morphism is the structure morphism for the affine space.*

Hence an smooth morphism is locally an étale morphism composed with the structure morphism of an affine space.

In order to prove this proposition, we need the following result, which was proven in the non-relative case in the previous lecture:

Lemma 3.2. *Let S be a scheme and n a non-negative integer. Consider $X = \mathbb{A}^n_S$ the affine space of dimension n over S. Then the $\mathcal{O}_S$-module $\Omega^1_{X/S}$ of relative differentials is free of rank n with basis $\{dx_1, \ldots, dx_n\}$.*

The differential $d : \mathcal{O}_X \to \Omega^1_{X/S}$ sends a local section s (a polynomial in the variables x_i with coefficients in $\mathcal{O}_S$) to $ds = \sum_{i=1}^n \frac{\partial s}{\partial x_i} dx_i$.

Proof. As we have seen in the previous lecture, $\Omega^1_{X/S}$ is generated as an $\mathcal{O}_X$-module by ds where s is a section of $\mathcal{O}_X$. Any section of $\mathcal{O}_X$ can be written as a polynomial in $x_1, \ldots, x_n$ with coefficients in $\mathcal{O}_S$. As d is a derivation $d : \mathcal{O}_X \to \Omega^1_{X/S}$, it is linear and satisfies the Leibniz rule, from where it follows easily that

$$ds = \sum_{i=0}^{n} \frac{\partial s}{\partial x_i} dx_i$$

holds, from what we can conclude that $\{dx_1, \ldots, dx_n\}$ generate $\Omega^1_{X/S}$ as an $\mathcal{O}_X$-module.

In order to prove that these set of generators form a basis, consider the free $\mathcal{O}_X$-module Ω' with basis $\{dx_1, \ldots, dx_n\}$ and the following derivation

$$d' : \mathcal{O}_X \to \Omega' : s \mapsto \sum_{i=0}^{n} \frac{\partial s}{\partial x_i} dx_i$$

By the universal property of $\Omega^1_{X/S}$, there exists a unique morphism of $\mathcal{O}_X$-modules $u : \Omega^1_{X/S} \to \Omega'$ such that $d' = u \circ d$, from which we deduce that u maps dx_i to dx_i for all i. Hence u is an isomorphism. □

Now we can prove Proposition 3.1.

Proof. From Proposition 2.10 we have that $\Omega^1_{X/Y}$ is locally free of finite type. Hence we can choose a neighbourhood U of x and sections $s_1, \dots, s_n \in \Gamma(\mathcal{O}_X, U)$ such that $\{ds_1, \dots, ds_n\}$ form a basis of $\Omega^1_{X/Y}|_U$. Consider the morphism $s = (s_1, \dots, s_n) : U \to \mathbb{A}^n_S = Y[t_1, \dots, t_n]$. Its composition with the structure morphism of $\mathbb{A}^n_S$

$$U \xrightarrow{s} \mathbb{A}^n_S \to Y$$

can be seen to coincide with $f|_U$.

It remains to prove that s is an étale morphism. By Proposition 2.13, it is enough with checking that $s^*\Omega^1_{\mathbb{A}^n_Y/Y} \to \Omega^1_{U/Y}$ is an isomorphism. We have that $\Omega^1_{\mathbb{A}^n_Y}$ is free and generated by $\{dt_1, \dots, dt_n\}$ (Lemma 3.2). Then, by definition of the map s, we have that $s^*\Omega^1_{\mathbb{A}^n_Y/Y}$ is generated by $\{ds_1, \dots, ds_n\}$, which provides the commutativity of the diagram

$$\begin{tikzcd} s^*\Omega^1_{\mathbb{A}^n_Y/Y} \arrow[r] \arrow[d, "\cong"] & \Omega^1_{U/Y} \arrow[d, "\cong"] \\ s^*\mathcal{O}^n_{\mathbb{A}^n_Y} \arrow[r, "\cong"] & \mathcal{O}^n_U \end{tikzcd}$$

Hence s is an étale morphism. $\square$

There is also a local model of étale morphisms with a more involved proof than the previous local characterization of smooth morphisms. We will not provide a proof here (it can be found in [4]), but we will state the result.

Proposition 3.3. *Let $f : X \to Y$ be a morphism of schemes. Consider x a point of X and denote $y = f(x)$. Assume f is étale at x. Then there exist open affine neighbourhoods $U = \mathrm{Spec}(B)$ of x and $V = \mathrm{Spec}(A)$ of y with $f(U) \subset V$, and a Y-immersion $U \to \mathbb{A}^1_Y$ that identifies U with an open subscheme of a closed subscheme $Z \subset \mathbb{A}^1_Y$ defined by a monic polynomial $P \in A[x]$ such that P' does not vanish on U.*

As for a morphism of schemes being smooth (resp. unramified, étale) is local on the source, we can define what it means for a morphism of schemes to be smooth (resp. unramified, étale) at a point.

Definition 3.4. Consider a morphism of schemes $f : X \to Y$ and let x be a point of X. We say that f is smooth (resp. unramified, étale) at the point x if there exists an open neighbourhood U of x in X such that $f|_U$ is smooth (resp. unramified, étale).

Proposition 3.5 (Jacobian criterion)**.** *Assume we have the following diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

where i is an immersion with ideal $\mathcal{I}$ and g is smooth. Given x a point of X, f is smooth at x if and only if there exists a finite number of sections $s_1, \dots, s_r$ of $\mathcal{I}$ on a neighbourhood of x such that:

1. *$\{s_1, \dots, s_r\}$ generate $\mathcal{I}_x$*

2. $\{(ds_1)(x),\dots,(ds_r)(x)\}$ *are linearly independent in* $\Omega^1_{Z/Y}\otimes k(x)$ *where* $k(x)$ *is the residue field at the point* x

Proof. We have that f is smooth at x if and only if $\mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{Z/Y}$ is an split injection on a neighbourhood of x (Proposition 2.10), and by Nakayama's lemma this holds if and only if there exist sections of $\mathcal{I}/\mathcal{I}^2$ in a neighbourhood of x that generate $\mathcal{I}/\mathcal{I}^2$ in that neighbourhood and have linearly independent images in $i^*\Omega^1_{Z/Y}\otimes k(x)$. But by the definition of the canonical morphism $\mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{Z/Y}$ (see [1, Proof of 1.1.26]) this is equivalent to the two properties above. $\square$

Proposition 3.6 (Implicit function theorem in algebraic geometry)**.** *Consider the following commutative diagram of morphisms of schemes*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

where i is an immersion with associated ideal $\mathcal{I}$ and both f and g are smooth at a point x of X.Then we can find an open neighbourhood U of $i(x)$ in Z, a non-negative integer n, an étale morphism $s : U \to \mathbb{A}^n_Y$ and a linear subspace V of $\mathbb{A}^n_Y$ such that we have a cartesian diagram

$$\begin{tikzcd} U\cap X \arrow[r, "i"] \arrow[d] & U \arrow[d, "s"] \\ V \arrow[r] & \mathbb{A}^n_Y \end{tikzcd}$$

Proof. By Proposition 3.5, we can take a set of sections $\{s_1,\dots,s_r\}$ of $\mathcal{I}$ generating $\mathcal{I}$ in a neighbourhood of x and such that $\{ds_1(x),\dots,ds_r(x)\}$ are linearly independent in $\Omega^1_{Z/Y}\otimes k(x)$. Then we can choose sections $\{s_{r+1},\dots,s_n\}$ of $\mathcal{O}_Z$ such that $\{ds_1(x),\dots,ds_r(x),ds_{r+1}(x),\dots,ds_n(x)\}$ form a basis of $\Omega^1_{Z/Y}\otimes k(x)$. Then sections $\{s_1,\dots,s_n\}$ define an étale S-morphism $s=(s_1,\dots,s_n): U \to \mathbb{A}^n_Y = Y[t_1,\dots,t_n]$ from an open neighbourhood U of x to $\mathbb{A}^n_Y$(see proof of Proposition 3.1) such that we have the following commutative diagram

$$\begin{tikzcd} U\cap X \arrow[r, "i"] \arrow[d] & U \arrow[d, "s"] \\ \mathbb{A}^{n-r}_Y \arrow[r] & \mathbb{A}^n_Y \end{tikzcd}$$

where $U\cap X$ is given by the inverse image by s of the linear subspace $V=\mathbb{A}^{n-r}_Y$ with equations $t_1=\dots=t_r=0$. $\square$

4 Regularity vs smoothness

Definition 4.1. An scheme X is said to be *regular* or *non-singular* if for every point $x\in X$ there exists an affine neighbourhood $U\subseteq X$ of x such that $\mathcal{O}_X(U)$ is a noetherian regular ring. Equivalently, X is regular if it is locally noetherian and for every point $x\in X$, the ring $\mathcal{O}_{X,x}$ is regular.

Recall that a noetherian ring R is said to be regular if every local ring $R_{\mathfrak{p}}$ is a regular ring, i.e. if for every local ring $R_{\mathfrak{p}}$ the maximal ideal can be generated by a regular sequence of parameters.

Remark 4.2. Observe that regularity is an absolute notion while smoothness is a relative notion.

Definition 4.3. A field k is said to be *perfect* if every field extension of k is separable over k.

Proposition 4.4. *If $f : X \to Y = \mathsf{Spec}(k)$ is an smooth morphism, where k is a field, then X is regular, and if x is a closed point of X, $k(x)$ is a finite separable extension of k, and the dimension of $\mathcal{O}_{X,x}$ is equal to the dimension $\dim_x(X)$ of the irreducible component of X containing x and to the rank of $\Omega^1_{X/Y}$.*

Conversely, if X is regular and k is perfect, then f is smooth.

Remark 4.5. If we choose in the previous setting k to be an imperfect ring, we can find an example of a regular scheme which is not smooth over its base. Indeed, take for example the field $\mathbb{F}_p(t)$ of rational functions. It is an imperfect field (see the question http://math.stackexchange.com/questions/106632/examples-of-fields-which-are-not-perfect): one can prove that the polynomial $x^p - t \in \mathbb{F}_p(t)[x]$ is irreducible and if one considers a root α of this polynomial in an extension of $\mathbb{F}_p(t)$ (thus $\alpha^p = t$), one gets $x^p - \alpha^p = (x-\alpha)^p$, so α is the unique root of $x^p - t$ in the extension, hence it is an inseparable polynomial. So any field extension k of $\mathbb{F}_p(t)$ containing α is inseparable. Then, if we consider the morphism of affine schemes $\mathsf{Spec}(k(x)) \to \mathsf{Spec}(\mathbb{F}_p(t))$, one can prove that this morphism is not smooth, while $\mathsf{Spec}(k(x))$ is regular (see http://mathoverflow.net/questions/12688/nonsingular-normal-schemes).

Below (in Proposition 4.8) we will give a more general statement relating smoothness of morphisms and regularity of schemes, but first we provide a reminder of some definitions.

Definition 4.6. We say a morphism $f : X \to Y$ of schemes is flat if $\mathcal{O}_{X,x}$ is an $\mathcal{O}_{Y,f(x)}$-flat module for any point $x \in X$.

Definition 4.7. Given a morphism of schemes $f : X \to Y$, a *geometric fibre* of f at a point $y \in Y$ is the scheme $X_y \coloneqq X \times_Y \mathsf{Spec}(\overline{k(y)})$ equipped with the reduced scheme structure, where $\overline{k(y)}$ is the algebraic closure of the residue field $k(y)$ in the point y.

Proposition 4.8. *Let $f : X \to Y$ be a morphism of finite presentation. Then the following are equivalent:*

1. *f is smooth*
2. *f is flat and the geometric fibres of f are regular schemes*

Proof. See [EGA IV, Corollaire (17.5.2)] □

5 Relative dimension

Let $f : X \to Y$ be a smooth morphism of schemes.

Definition 5.1. Let $x \in X$ be a point of X. We call *relative dimension* of f at x the integer

$$\dim_x(f) \coloneqq \dim_{k(x)} \Omega^1_{X/Y} \otimes k(x) = \mathrm{rg}_{\mathcal{O}_{X,x}} \Omega^1_{X/Y,x}$$

In particular this dimension coincides with the dimension of the irreducible component of the fibre $X_{f(x)}$ that contains x [EGA IV, p. 17.10.2].

Remark 5.2. As f is smooth, we have by Proposition 2.10 that $\Omega^1_{X/Y}$ is locally free of finite type, thus relative dimension of f is a locally constant function on x.

Also as a consequence of Proposition 2.10 and Proposition 2.13, the smooth morphism f is étale if and only if $\Omega^1_{X/Y} = 0$ which happens if and only if $\dim_x(f) = 0$ for every point $x \in X$. Hence by the geometrical interpretation of the relative dimension at the end of Definition 5.1 and by Proposition 4.8, we have that a morphism of schemes is étale if and only if it is locally of finite presentation, unramified and flat.

Definition 5.3. Given a smooth morphism $f : X \to Y$ of schemes, we say that f is of *pure relative dimension* r if it is of constant locally relative dimension r.

Proposition 5.4. *If $f : X \to Y$ is a smooth morphism of schemes of relative dimension r, then the de Rham complex $\Omega^\bullet_{X/Y}$ is zero in degrees strictly bigger than r, and $\Omega^i_{X/Y}$ is a locally free $\mathcal{O}_X$-mopdule of rank $\binom{r}{i}$. In particular $\Omega^r_{X/Y}$ is an invertible $\mathcal{O}_X$-module.*

Proof. It is easily deduced from the construction of the de Rham complex, which was done in the previous lecture. □

References

[1] François Charles. *18.726: the de Rham complex and topics in deformation theory*. English. 2014. URL: `http://www.math.u-psud.fr/~fcharles/CoursAG.pdf`.

[EGA IV] A. Grothendieck. *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV.* French. 32. 1967, p. 361.

[2] Luc Illusie. *Complexe cotangent et déformations. I.* French. Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971, pp. xv+355.

[3] Luc Illusie. “Frobenius and Hodge degeneration”. English. In: José Bertin et al. *Introduction to Hodge theory.* American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2002, pp. 99–149.

[4] Michel Raynaud. *Anneaux locaux henséliens.* French. Lecture Notes in Mathematics, Vol. 169. Springer-Verlag, Berlin-New York, 1970, pp. v+129.

[5] Charles A. Weibel. *An introduction to homological algebra.* English. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450.