# Cayley–Hamilton algebras

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This is a basic introduction to Cayley–Hamilton algebras in the spirit of [Le 07].

## 1 Definition

By "algebra" we mean a unital associative $\mathbb{C}$-algebra.

**Definition 1.** An *algebra with trace* $(R, \mathrm{tr})$ is an algebra $R$ together with a $\mathbb{C}$-linear function $\mathrm{tr} : R \to R$, satisfying the following properties for all $a, b \in R$.

1. *(Maps into center)* $\mathrm{tr}(a)b = \mathrm{tr}(b)a$,
2. *(Necklace property)* $\mathrm{tr}(ab) = \mathrm{tr}(ba)$,
3. *(Linear w.r.t. traces)* $\mathrm{tr}(\mathrm{tr}(a)b) = \mathrm{tr}(a)\mathrm{tr}(b)$.

An algebra with trace $(R, \mathrm{tr})$ is called *Cayley–Hamilton* of degree $n$ if for all $a \in R$ it additionally satisfies:

1. $\mathrm{tr}(1) = n$,
2. $\chi_a^{(n)}(a) = 0$.

The *characteristic polynomial* $\chi_a^{(n)}$ of $a$ is defined as follows. If $a$ would have been an $n \times n$-matrix, then the characteristic polynomial has as coefficients symmetric polynomials in $n$ variables $x_1, \ldots, x_n$ (the eigenvalues of $a$). These symmetric polynomials are freely generated as a ring by the power sum polynomials $x_1^k + \cdots + x_n^k = \mathrm{tr}(a^k)$. This fact enables us to interpret each coefficient in a unique way in terms of traces. For example,

$$\chi_a^{(2)}(t) = t^2 - \mathrm{tr}(a)t + \frac{1}{2}\left(\mathrm{tr}(a)^2 - \mathrm{tr}(a^2)\right).$$

Algebras with trace can be made into a category $\mathsf{alg@}$ with as arrows the trace preserving algebra maps. The full subcategory of degree $n$ Cayley–Hamilton algebras is denoted by $\mathsf{alg@n}$.

For a Cayley–Hamilton algebra $R$, the image $\mathrm{tr}(R) \subseteq R$ is a subalgebra of the center of $R$: it is clearly a linear subspace, and $\mathrm{tr}(a)\mathrm{tr}(b) = \mathrm{tr}(\frac{1}{n}\mathrm{tr}(a)\mathrm{tr}(b))$, which shows that the product of two elements in $\mathrm{tr}(R)$ again lies in $\mathrm{tr}(R)$. Often $\mathrm{tr}(c) = nc$ for elements $c$ in the center $Z(R)$ of $R$, and in this case $\mathrm{tr}(R) = Z(R)$.

**Example 2** (Orders)**.** A *matrix algebra* $\mathrm{M}_n(C)$ over a commutative ring $C$ is a Cayley–Hamilton algebra of degree $n$ in a natural way. As trace map tr we take the map sending a matrix to the sum of its diagonal elements (surprise!). Now use the Cayley–Hamilton theorem.

An *Azumaya algebra* $A$ over $C$ of rank $n^2$ is étale locally a matrix algebra as above. Using the necklace property, we get that $\mathrm{tr}(aba^{-1}) = \mathrm{tr}(b)$. But this means that tr descends to a map $\mathrm{tr} : A \to A$. It can be checked locally that this map satisfies all necessary properties. In particular each $a \in A$ is a zero of its own Cayley–Hamilton polynomial.

An *order* $R$ over a commutative integrally closed domain $C$ can be embedded into an Azumaya algebra, for example in the central simple algebra $A = R \otimes_C K$, where $K$ is the fraction field of $C$. This yields a trace map $\mathrm{tr} : R \to K$ and it remains to show that $\mathrm{tr}(R) \subseteq C$. We know that $\mathrm{tr}(R)$ is a subalgebra of $K$ containing $C$. It is moreover finitely generated as a $C$-module, because $R$ is finitely generated as $C$-module. But finite ring extensions are integral [Stacks, Tag 00GH], so from the fact that $C$ is integrally closed it follows that $\mathrm{tr}(R) = C$.

**Example 3** (Free Cayley–Hamilton algebras)**.** To each algebra $R$, we can associate a universal trace map $\pi : R \to R/[R, R]$. Here $[R, R]$ is the linear subspace of commutators $\{ab - ba \mid a, b \in R\} \subseteq R$. We use the notation $\oint R$ for the free commutative ring $\mathrm{Sym}\,(R/[R, R])$ and we write $\int R = R \otimes \oint R$. The universal trace map gives a trace map on $\int R$ via $\mathrm{tr}(r \otimes t) = \pi(r)t$. It is easy to see that $\int : \mathsf{alg} \to \mathsf{alg@}$ is the free functor, i.e. the left adjoint to the forgetful functor.

We can then construct a free degree $n$ Cayley–Hamilton algebra on $R$ by considering $\int R$ and adding the necessary relations $\mathrm{tr}(1) = n$ and $\chi_a^{(n)}(a) = 0$. We use the notation $\int_n R$ for the resulting algebra, so $\int_n : \mathsf{alg} \to \mathsf{alg@n}$ is the free functor.

Note that $R/[R, R]$ is also known as the Hochschild homology $\mathrm{HH}_0(R, R)$.

## 2 Trace preserving representations

For an algebra $R$, there is a scheme $\mathrm{rep}_n(R)$ with as $D$-points the n-dimensional representations $R \to \mathrm{M}_n(D)$. If $R$ comes equipped with a trace map, then we can consider the closed subscheme $\mathrm{trep}_n(R)$ with as $D$-points the *trace preserving* representations $R \to \mathrm{M}_n(D)$. Because the $\mathrm{PGL}_n$-action on $\mathrm{M}_n$ is trace preserving, the $\mathrm{PGL}_n$-action on $\mathrm{rep}_n$ restricts to a $\mathrm{PGL}_n$-action on $\mathrm{trep}_n(R)$.

Using the adjunction between free and forgetful functor, we see that the two schemes are related by

$$\mathrm{rep}_n(R) = \mathrm{trep}_n\left(\int R\right) = \mathrm{trep}_n\left(\int_n R\right).$$

Moreover, to each $\mathrm{PGL}_n$-scheme $X$, we can associate its *witness algebra*

$$\begin{aligned} \Uparrow^n(X) &= \mathrm{M}_n(\Gamma(X, \mathcal{O}_X))^{\mathrm{PGL}_n} \\ &= \{\text{equivariant morphisms } X \to \mathrm{M}_n(\mathbb{C})\}. \end{aligned}$$

Here $\mathrm{M}_n(\mathbb{C})$ is of course just $\mathbb{A}^{n^2}$ as a scheme, but the suggestive notation makes the $\mathrm{PGL}_n$-action (by matrix conjugation) more clear.

The witness algebra comes equipped with a trace map, inherited from the matrix algebra $\mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$. If we interpret elements of $\Uparrow^n(X)$ as equivariant morphisms $f : X \to \mathrm{M}_n$, then the trace map is given by

$$\operatorname{tr}(f)(x) = \operatorname{tr}(f(x)).$$

It is easy to check that this trace is again equivariant map. More precisely, the traces of the witness algebra are precisely given by the diagonal matrices $\Gamma(X, \mathcal{O}_X)^{\mathrm{PGL}_n}$ (i.e. the diagonal matrices in the witness algebra). Now it is clear that the witness algebra is a Cayley–Hamilton algebra of degree $n$.

**Theorem 4** (Procesi)**.** *The functor $\Uparrow^n$ is left inverse to $\operatorname{trep}_n$ in the diagram*

$$\begin{tikzcd} \mathsf{alg@n} \arrow[rr, bend left, "\operatorname{trep}_n"] & & \mathsf{PGL_n\text{-}schemes.} \arrow[ll, bend left, "\Uparrow^n"] \end{tikzcd} \tag{1}$$

*Proof.* See [Le 07] or Procesi's paper [Pro87]. ☐

As an immediate consequence we find that $\int_n R = \Uparrow^n(\operatorname{rep}_n R)$. Some other useful corollaries are the following from [De +05].

**Corollary 5.**

1. *If $(R, \operatorname{tr})$ is in* alg@n*, then $(R, k \cdot \operatorname{tr})$ is in* alg@kn*.*
2. *If $(R, \operatorname{tr}_R)$ is in* alg@n *and $(S, \operatorname{tr}_S)$ is in* alg@m*, then $(R \otimes S, \operatorname{tr}_R \otimes \operatorname{tr}_S)$ is in* alg@nm*.*
3. *If $(R, \operatorname{tr}_R)$ and $(S, \operatorname{tr}_S)$ are in* alg@n*, then $(R \times S, \operatorname{tr}_R \times \operatorname{tr}_S)$ is in* alg@n*.*

*Proof.* This is easy for $R$ and $S$ matrix algebras over a commutative ring. But by Theorem 4, any Cayley–Hamilton algebra inherits its trace from such a matrix algebra. For details, see [De +05]. ☐

# 3 Orders

We already saw that orders are Cayley-Hamilton algebras in a natural way. Now we will try to specify which Cayley-Hamilton algebras come from orders.

**Lemma 6.** *Let $R$ be a Cayley–Hamilton algebra of degree $n$. The following are equivalent.*

1. *$\operatorname{trep}_n(R)$ is a $\mathrm{PGL}_n$-torsor.*
2. *$R$ is an Azumaya algebra of constant rank $n^2$.*

*Proof.* Both $\mathrm{PGL}_n$-torsors over $\operatorname{Spec}(C)$ and Azumaya algebras over $C$ are classified by $\mathrm{H}^1(C, \mathrm{PGL}_n)$ and the correspondence is given by

$$A \mapsto \operatorname{trep}_n(A) = \operatorname{rep}_n(A),$$

see [Le 10]. ☐

Note that the functor $\mathrm{trep}_n : \mathsf{alg@n}^{\mathrm{op}} \to \mathsf{schemes}$ has a left adjoint, given by sending a scheme $X$ to the Cayley–Hamilton algebra $\mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$. This implies that $\mathrm{trep}_n$ sends colimits in $\mathsf{alg@n}$ to limits in $\mathsf{schemes}$. For example, one can check that the diagram

$$\begin{tikzcd} R \arrow[r] & R \otimes_C K \\ C \arrow[u] \arrow[r] & K \arrow[u] \end{tikzcd} \tag{2}$$

is a pushout in $\mathsf{alg@n}$, for $R$ a Cayley–Hamilton algebra for which $\mathrm{tr}(R) = C$ is a domain with fraction field $K$. Applying $\mathrm{trep}_n$ then gives a pullback diagram of schemes

$$\begin{tikzcd} \mathrm{trep}_n(R \otimes_C K) \arrow[r] \arrow[d] & \mathrm{trep}_n(R) \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(C) \end{tikzcd} \tag{3}$$

But if $R$ is an order, then $R \otimes_C K$ is a central simple algebra, in other words $\mathrm{trep}_n(R \otimes_C K)$ is a $\mathrm{PGL}_n$-torsor. So the $\mathrm{PGL}_n$-schemes corresponding to orders stand out by generically being a $\mathrm{PGL}_n$-torsor. For the converse, suppose that $C$ is integrally closed. If $\mathrm{trep}_n(R \otimes_C K)$ is a $\mathrm{PGL}_n$-torsor then $R \otimes_C K$ is a central simple algebra, so $R$ is an order.

## References

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