Spectral sequences: an Introduction

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In these notes we introduce the notion of a spectral sequence and give some basic properties. Even from the definition of a spectral sequence itself, things could appear very technical, so some of the proofs and details will be omitted. For example, the Complete Convergence Theorem will not be studied in detail, as it is not directly needed. On the other hand, some known properties like the Universal Coefficient Theorem will be shown using spectral sequences in order to motivate their use and to get more familiar with the subject. We basically follow [2, Ch. 5], but also recommend [1, Ch. XI].

1 Terminology

Definition 1.1 A homology spectral sequence (starting with E^a, $a \geq 0$) in an abelian category $\mathcal{A}$ consist of the following data:

1. A family $\{E^r_{pq}\}$ of objects of $\mathcal{A}$ defined for all integers p, q and $r \geq a$;
2. Maps $d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$ that are differentials in the sense that $d^r d^r = 0$, so that the lines of slope $(1-r)/r$ in the lattice E^r_{**} form chain complexes in $\mathcal{A}$;
3. Isomorphisms between E^{r+1}_{pq} and the homology of E^r_{**} at the spot E^r_{pq}:

$$E^{r+1}_{pq} \cong \mathrm{Ker}(d^r_{pq})/\mathrm{Im}(d^r_{p+r,q-r+1}).$$

The total degree of the term E^r_{pq} is $n = p+q$, so the terms of total degree n lie on a line of slope -1, and each differential d^r_{pq} decreases the total degree by one.
A morphism $f : E \to E'$ between two homology spectral sequences is a family of maps $f^r_{pq} : E^r_{pq} \to E'^r_{pq}$ in $\mathcal{A}$ (for r suitably large) with $d^r f^r = f^r d^r$ and such that each f^{r+1}_{pq} is the map induced by f^r_{pq} on homology. It follows easily that there exists a category of homology spectral sequences in any abelian category $\mathcal{A}$.

$E^{r+1}_{pq} \cong \mathrm{Ker}(d^r_{pq})/\mathrm{Im}(d^r_{p+r,q-r+1})$ is a subquotient of E^r_{pq}, so we obtain a nested family of subobjects of E^a_{pq}:

$$0 = B^a_{pq} \subseteq \cdots \subseteq B^r_{pq} \subseteq B^{r+1}_{pq} \subseteq \cdots \subseteq Z^{r+1}_{pq} \subseteq Z^r_{pq} \subseteq \cdots \subseteq Z^a_{pq} = E^a_{pq}$$

such that $E^r_{pq} \cong Z^r_{pq}/B^r_{pq}$. Indeed, for $r > a$ consider

$$Z^r_{p+r,q-r+1}/B^r_{p+r,q-r+1} \xrightarrow{d^r_{p+r,q-r+1}} Z^r_{pq}/B^r_{pq} \xrightarrow{d^r_{pq}} Z^r_{p-r,q+r-1}/B^r_{p-r,q+r-1}.$$

Then $E^{r+1}_{pq} \cong Z^{r+1}_{pq}/B^{r+1}_{pq}$, where $Z^{r+1}_{pq}/B^r_{pq} = \mathrm{Ker}(d^r_{pq})$, $B^{r+1}_{pq}/B^r_{pq} = \mathrm{Im}(d^r_{p+r,q-r+1})$ and $B^{r+1}_{pq} \subseteq Z^{r+1}_{pq}$, by which

$$B^r_{pq} \subseteq B^{r+1}_{pq} \subseteq Z^{r+1}_{pq} \subseteq Z^r_{pq}.$$

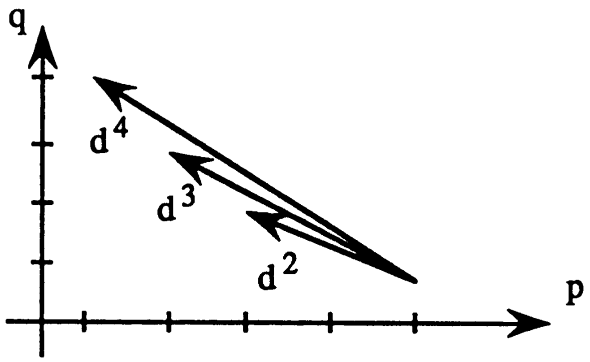

Figure 1: The differentials of a homology spectral sequence.

If our abelian category $\mathcal{A}$ is complete and cocomplete (e.g. $\mathcal{A} = {}_R\mathcal{M}$), we introduce the intermediate objects

$$B^{\infty}_{pq} = \bigcup_{r=a}^{\infty} B^r_{pq} \quad \text{and} \quad Z^{\infty}_{pq} = \bigcap_{r=a}^{\infty} Z^r_{pq}$$

and define $E^{\infty}_{pq} = Z^{\infty}_{pq}/B^{\infty}_{pq}$. We say that the spectral sequence abuts to E^{∞}_{pq}. We consider the terms E^r_{pq} of the spectral sequence as successive approximations (via successive formation of subquotients) to E^{∞}_{pq}.

Example 1.2 A first quadrant (homology) spectral sequence is one with $E^r_{pq} = 0$ unless $p \geq 0$ and $q \geq 0$, that is, the point (p,q) belongs to the first quadrant of the plane. At the r-stage, the differentials d^r_{**} go r columns to the left and $r-1$ rows up, so for suitably large r all differentials d^s_{**} for $s \geq r$ leaving and ending at the (p,q)-spot are zero (take for example $r > (p+1) \vee (q+2)$). This means that E^r_{pq} is ultimately constant in r and for this stabilized value we clearly have that it equals E^{∞}_{pq}.

We also have a cohomological analogue:

Definition 1.3 A cohomology spectral sequence (starting with E_a, $a \geq 0$) in an abelian category $\mathcal{A}$ is a family $\{E^{pq}_r\}$ of objects ($r \geq a, p, q \in \mathbb{Z}$), together with maps d^{pq}_r:

$$d^{pq}_r : E^{pq}_r \to E^{p+r,q-r+1}_r,$$

which are differentials and give isomorphisms between E_{r+1} and the homology of E_r.

For such a spectral sequence, the differential d^{**}_r increases the total degree $p+q$ of E^{pq}_r by one and one defines the objects $B^{pq}_{\infty}, Z^{pq}_{\infty}$ and E^{pq}_{∞} analogously.

Let's consider a special type of spectral sequences, namely the bounded ones.

Definition 1.4 A homology spectral sequence is said to be bounded if for each n there are only finitely many nonzero terms of total degree n in E^a_{**}.

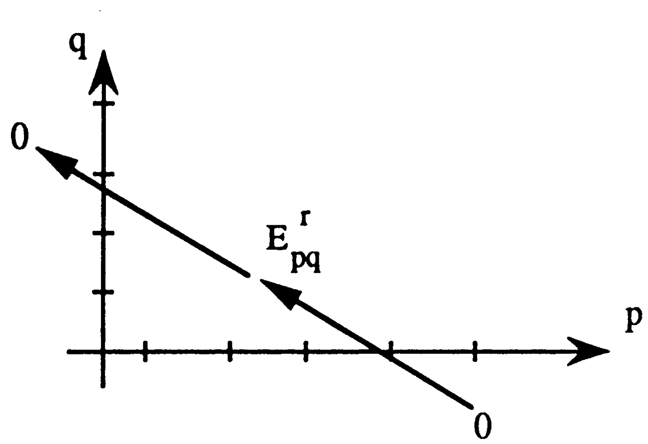

Figure 2: d^r_{pq} ultimately becomes zero.

Because d^r_{**} decreases the total degree by one this means that for all p and q there is an r_0 such that $E^r_{pq} = E^{r+1}_{pq}$ for all $r \geq r_0$. As in the example of a first quadrant spectral sequence this stable value equals E^∞_{pq}. For such spectral sequences we introduce the notion of convergence

Definition 1.5 A bounded spectral sequence $\{E^r_{pq}\}$ converges to H_* if there exists a family of objects H_n of $\mathcal{A}$, each having a finite filtration

$$0 = F_s H_n \subseteq \cdots \subseteq F_{p-1} H_n \subseteq F_p H_n \subseteq F_{p+1} H_n \subseteq \cdots \subseteq F_t H_n = H_n,$$

such that $E^\infty_{pq} \cong F_p H_{p+q} / F_{p-1} H_{p+q}$. We write down this bounded convergence by

$$E^a_{pq} \Rightarrow H_{p+q}.$$

Dually, a cohomology spectral sequence is called bounded if there are only finitely many nonzero terms in each total degree in E^{**}_a and such a spectral sequence converges to H^* if there is a finite filtration for each n

$$0 = F^t H^n \subseteq \cdots \subseteq F^{p+1} H^n \subseteq F^p H^n \subseteq \cdots \subseteq F^s H^n = H^n,$$

with

$$E^{pq}_\infty \cong F^p H^{p+q} / F^{p+1} H^{p+q}.$$

Example 1.6 Let $\{E^r_{pq}\}$ be a first quadrant homology spectral sequence, which converges to H_*. Because for any n we have

$$E^\infty_{p,n-p} = 0 \quad \text{for } p \leq 0 \text{ or } p > n,$$

H_n has a finite filtration of length $n+1$:

$$0 = F_{-1} H_n \subseteq F_0 H_n \subseteq \cdots \subseteq F_{n-1} H_n \subseteq F_n H_n = H_n.$$

The bottom piece $F_0H_n = E^\infty_{0n}$ is located on the y-axis, and the top quotient $H_n/F_{n-1}H_n \cong E^\infty_{n0}$ is located on the x-axis. Since each arrow landing on the x-axis is zero, and each arrow leaving the y-axis is zero, each E^∞_{n0} is a subobject of E^a_{n0} and each E^∞_{0n} is a quotient of E^a_{0n}. The terms E^r_{n0} on the x-axis are called the base terms and the terms E^r_{0n} on the y-axis are called the fiber terms. The resulting maps

$$H_n \to E^\infty_{n0} \subseteq E^a_{n0},$$

and

$$E^a_{0n} \to E^\infty_{0n} \subseteq H_n$$

are known as the edge homomorphisms of the spectral sequence.

Definition 1.7 A spectral sequence collapses at E^r $(r \geq 2)$ if there is exactly one nonzero row or column in the lattice E^r_{**}.

If a bounded collapsing spectral sequence converges to H_*, we can read the H_n off: it is the unique nonzero E^r_{pq} with $p+q=n$. By this, we are highly interested in collapsing spectral sequences at stage 1 or 2. Moreover, we say that a spectral sequence $\{E^r_{pq}\}$ $(r \geq a)$ degenerates at sheet r if for all $s \geq r$ the differentials d^s_{**} are all zero. This means of course that $E^r_{pq} = E^\infty_{pq}$. Obviously, there's a dual notion for cohomology spectral sequences.

Another application will lie in exact sequences some special spectral sequences offer.

Lemma 1.8 (2 columns) *For a bounded homology spectral sequence converging to H_* and having $E^2_{pq} = 0$ unless $p = 0$ or 1, there are short exact sequences*

$$0 \to E^2_{0n} \to H_n \to E^2_{1,n-1} \to 0.$$

Proof. Because the differentials d^s_{**} go s columns to the left and $s-1$ rows up, we see that the spectral sequence degenerates at $r = 2$, whence

$$E^\infty_{pq} = E^2_{pq}.$$

Fix n. By definition of convergence, for all $p \neq 0,1$

$$0 = E^\infty_{p,n-p} \cong F_pH_n/F_{p-1}H_n \Rightarrow F_pH_n = F_{p-1}H_n, \tag{1}$$

by which $E^\infty_{0,n} = F_0H_n$, because by (1) $F_{-1}H_n = F_{-2}H_n = \cdots = 0$.
This implies

$$E^\infty_{1,n-1} \cong H_n/E^\infty_{0,n}. \tag{2}$$

Define

$$\begin{tikzcd} E^2_{0n} \arrow[r, "="] \arrow[rr, bend left, "f"] & E^\infty_{0n} \arrow[r, hook] & H_n \end{tikzcd}$$

and

$$\begin{tikzcd} H_n \arrow[r, two heads] \arrow[rr, bend right, "g"'] & E^\infty_{1,n-1} \arrow[r, hook] & E^2_{1,n-1} \end{tikzcd}$$

By (2) the sequence

$$0 \to E^2_{0n} \xrightarrow{f} H_n \xrightarrow{g} E^2_{1,n-1} \to 0$$

is indeed exact (In fact, the maps f and g are just edge homomorphisms). $\square$

There is a dual statement for cohomolgy spectral sequences in which the arrows are reversed. We will state the following similar lemma in this framework.

Lemma 1.9 (2 rows) *For a bounded cohomology spectral sequence converging to H^* and having $E_2^{pq} = 0$ unless $q = 0$ or $r > 0$, there is a long exact sequence*

$$\cdots \to H^n \to E_2^{n-r,r} \xrightarrow{d_{r+1}} E_2^{n+1,0} \to H^{n+1} \to \cdots$$

In practice the spectral sequences that pop up aren't always bounded, so one has to introduce more general types of such sequences. To begin with

Definition 1.10 A homology spectral sequence is said to be bounded below if for each n there is an integer $s = s(n)$ such that the terms E^a_{pq} of total degree n vanish for all $p < s$.
Dually, a cohomology spectral sequence is said to be bounded below if for each n the terms of total degree n vanish for large p. Moreover, the dual notion of bounded above also exists in both frameworks.

Clearly bounded spectral sequences are also bounded below. Right half-plane homology spectral sequences are bounded below but not bounded. Dually, left half-plane cohomology spectral sequences are bounded below but not bounded.

In general, one can define convergence of a spectral sequence, but we will not go deeper in this as it requires some technical notions.

2 Spectral Sequence of a Filtration and Convergence Theorem

Let C_* be a chain complex in an abelian category $\mathcal{A}$. By a filtration F of C we mean an ordered family of chain subcomplexes $\cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq \cdots$ of C. We have the following

Theorem 2.1 (Construction Theorem) *A filtration F of a chain complex C naturally determines a spectral sequence starting with $E^0_{pq} = F_pC_{p+q}/F_{p-1}C_{p+q}$ and $E^1_{pq} = H_q(E^0_{p*})$. Moreover, the differentials are induced by the differential of C.*

Again, we have special types of filtrations: we call a filtration bounded if for each n there are integers $s < t$ such that $F_sC_n = 0$ and $F_tC_n = C_n$. In this case there are only finitely many nonzero terms of total degree n in E^0_{**}, so the associated spectral sequence is bounded. Subsequently, a filtration on C is called bounded below if for each n there is an integer $s = s(n)$ such that $F_sC_n = 0$, and it is called bounded above if for each n there is a $t = t(n)$ such that $F_tC_n = C_n$. A filtration is exhaustive if $C = \cup F_pC$, so we see that bounded above implies exhaustive. Finally, a filtration is called canonically bounded if $F_{-1}C = 0$ and $F_nC_n = C_n$ for each n. For the associated spectral sequence it holds $E^0_{pq} = F_pC_{p+q}/F_{p-1}C_{p+q}$. Bounded below and exhaustivity are highly appreciated properties, because we have the following theorem:

Theorem 2.2 (Classical Convergence Theorem) *1. Suppose that the filtration on C is bounded. Then the spectral sequence is bounded and converges to $H_*(C)$:*

$$E^1_{pq} = H_q(F_pC/F_{p-1}C) \Rightarrow H_{p+q}(C);$$

2. Suppose that the filtration on C is bounded below and exhaustive. Then the spectral sequence is bounded below and also converges to $H_*(C)$.

Moreover, the convergence is natural in the sense that if $f: C \to C'$ is a map of filtered complexes, then the map $f_*: H_*(C) \to H_*(C')$ is compatible with the corresponding map of spectral sequences.

3 Spectral sequences of a double complex

Definition 3.1 Let $\mathcal{A}$ be an abelian category. A bicomplex or double chain complex $C = C_{**}$ in $\mathcal{A}$ is a collection of objects $C_{p,q}$ of $\mathcal{A}$ indexed by two integers p and q together with a horizontal differential $d^h: C_{pq} \to C_{p-1,q}$ and a vertical differential $d^v: C_{pq} \to C_{p,q-1}$

$$\begin{tikzcd} C_{p-1,q} \arrow[d, "d^v"'] & C_{pq} \arrow[l, "d^h"'] \arrow[d, "d^v"] \\ C_{p-1,q-1} & C_{p,q-1} \arrow[l, "d^h"] \end{tikzcd}$$

satisfying the following identities

$$d^v d^v = d^h d^h = d^v d^h + d^h d^v = 0.$$

Associated to a bicomplex there is a product total complex defined by

$$(\mathrm{Tot}^\Pi C_{**})_n := \Pi_{p+q=n} C_{pq}$$

with differential $d = d^h + d^v$. The homology groups $H_n(\mathrm{Tot}^\Pi C)$ are called the homology groups of the bicomplex C.

There are two filtrations associated to every double (chain) complex $C = C_{**}$, resulting in two spectral sequences related to the homology of $\mathrm{Tot}(C)$. Playing these spectral sequences off against each other will come in handy in calculating homology. Evidently, all the following dualizes for double cochain complexes.
Firstly, we can filter the (product or direct sum) total complex $\mathrm{Tot}(C)$ by the columns of C, denoting by ${}^IF_n\mathrm{Tot}(C)$ the total complex of the double subcomplex

$$({}^IC_{\tau\le n})_{pq} = \begin{cases} C_{pq} & \text{if } p \le n \\ 0 & \text{if } p > n \end{cases} \qquad \begin{array}{ccc|cc} \cdots & * & * & 0 & 0 \\ \cdots & * & * & 0 & 0 \\ \cdots & * & * & 0 & 0 \\ \cdots & * & * & 0 & 0 \end{array}$$

of C. From the Construction Theorem this gives rise to a spectral sequence $\{{}^IE^r_{pq}\}$, starting with

$$\begin{aligned} {}^IE^0_{pq} &= \frac{{}^IF_p\mathrm{Tot}(C)_{p+q}}{{}^IF_{p-1}\mathrm{Tot}(C)_{p+q}} = \frac{\bigoplus_{k+l=p+q}({}^IC_{\tau\le p})_{kl}}{\bigoplus_{k+l=p+q}({}^IC_{\tau\le p-1})_{kl}} \\ &= \frac{\bigoplus_{k\le p} C_{k,n-k}}{\bigoplus_{k\le p-1} C_{k,n-k}} = C_{pq}. \end{aligned}$$

The maps $d^0_{pq} : C_{pq} \to C_{p,q-1}$ are induced by the differential of $\mathrm{Tot}(C)$, hence equals the vertical differential d^v of C, whence

$$ {}^I E^1_{pq} = H^v_q(C_{p*}). $$

Subsequently, the maps $d^1 : H^v_q(C_{p*}) \to H^v_q(C_{p-1,*})$ are induced on homology from the horizontal differentials d^h of C, so (suggestive notation)

$$ {}^I E^2_{pq} = H^h_p H^v_q(C). $$

If C is a first quadrant double complex, the filtration is canonically bounded, and we have by the Classical Convergence Theorem the convergent spectral sequence

$$ {}^I E^2_{pq} = H^h_p H^v_q(C) \Rightarrow H_{p+q}(\mathrm{Tot}(C)). $$

Secondly, we filter by the rows, denoting by ${}^{II}F_n\mathrm{Tot}(C)$ the total complex of

$$ ({}^{II}C_{\tau\leq n})_{pq} = \begin{cases} C_{pq} & \text{if } q \leq n \\ 0 & \text{if } q > n \end{cases} \qquad \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline * & * & * & * & * & * \\ * & * & * & * & * & * \end{array} $$

In this case the associated spectral sequence $\{{}^{II}E^r_{pq}\}$ starts with

$$ {}^{II}E^0_{pq} = \frac{{}^{II}F_p\mathrm{Tot}(C)_{p+q}}{{}^{II}F_{p-1}\mathrm{Tot}(C)_{p+q}} = \frac{\bigoplus_{k\leq p} C_{n-k,k}}{\bigoplus_{k\leq p-1} C_{n-k,k}} = C_{qp}. $$

Observe the interchange of p and q! Hence $d^0 = d^h$, so ${}^{II}E^1_{pq} = H^h_q(C_{*p})$. The maps d^1 are induced from the vertical differentials d^v of C, so

$$ {}^{II}E^2_{pq} = H^v_p H^h_q(C). $$

As before, if C is a first quadrant double complex, this filtration is canonically bounded, and the spectral sequence converges to $H_*\mathrm{Tot}(C)$. Hence such types of double complexes are highly attractive, for they provide two spectral sequences converging to the same objects.

Lemma 3.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ an exact covariant functor. If (A, δ) is a (co)chain complex in $\mathcal{A}$, then*

$$ H_*F(A) \cong F(H_*(A)), $$

respectively

$$ H^*F(A) \cong F(H^*(A)). $$

Proof. Let's proof the lemma for a cochain complex

$$ \cdots \to A_{n-1} \xrightarrow{\delta_{n-1}} A_n \to \cdots $$

By exactness of F, the sequences

$$ 0 \to F(\mathrm{Ker}(f)) \to A \xrightarrow{F(f)} B \quad \text{and} \quad F(A) \xrightarrow{F(f)} F(\mathrm{Im}(f)) \to 0 $$

are exact for a morphism $f : A \to B$, whence

$$\mathrm{Ker}(F(f)) \cong F(\mathrm{Ker}(f)) \quad \text{and} \quad \mathrm{Im}(F(f)) \cong F(\mathrm{Im}(f))$$

From this we obtain

$$H^*F(A) = \frac{\mathrm{Ker}(F(\delta_n))}{\mathrm{Im}(F(\delta_{n-1}))} \cong \frac{F(\mathrm{Ker}(\delta_n))}{F(\mathrm{Im}(\delta_{n-1}))}.$$

The following short exact sequence

$$0 \to \mathrm{Im}(\delta_{n-1}) \to \mathrm{Ker}(\delta_n) \to H^n(A) \to 0$$

yields

$$F(H^n(A)) \cong F(\mathrm{Ker}(\delta_n))/F(\mathrm{Im}(\delta_{n-1})),$$

finishing the proof. The proof for chain complexes is exactly the same. □

The same result holds for contravariant functors.

Proposition 3.3 (Künneth spectral sequence) *Let P_* be a bounded below chain complex of projective left R-modules and M an R-module. Then there is a boundedly converging right half-plane cohomology spectral sequence*

$$E_2^{pq} = \mathrm{Ext}_R^p(H_q(P), M) \Rightarrow H^{p+q}({}_R\mathrm{Hom}(P, M)).$$

Proof. Let $M \to Q^*$ be an injective resolution and consider the upper half-plane double cochain complex ${}_R\mathrm{Hom}(P, Q)$, which looks like

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & {}_R\mathrm{Hom}(P^{-p+1}, Q^{q+1}) & \longrightarrow & {}_R\mathrm{Hom}(P^{-p}, Q^{q+1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & {}_R\mathrm{Hom}(P^{-p+1}, Q^{q}) & \longrightarrow & {}_R\mathrm{Hom}(P^{-p}, Q^{q}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

(recall the sign convention $P^{-p} = P_p$). The first filtration is complete, bounded above (P_* bounded below), whence regular and exhaustive so the conditions of the Complete Convergence Theorem are satisfied. Moreover,

$${}^I E_1^{pq} = H^q({}_R\mathrm{Hom}(P^{-p}, Q)) \cong {}_R\mathrm{Hom}(P^{-p}, H^q(Q)),$$

where we used projectivity of P^{-p} and Lemma 3.2. So

$${}^I E_2^{pq} = \begin{cases} H^p({}_R\mathrm{Hom}(P, M)) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

This means that the spectral sequence collapses at $r = 2$ and

$$H^p({}_R\mathrm{Hom}(P, M)) \cong H^p({}_R\mathrm{Hom}(P, Q)).$$

The second filtration is bounded below and exhaustive for the direct sum total complex. Since Q^q is injective, the contravariant analogue ofLemma 3.2 yields

$${}^{II}E_1^{pq} = H^q({}_R\mathrm{Hom}(P_*, Q^p)) \cong {}_R\mathrm{Hom}(H_q(P_*), Q^p).$$

Hence

$${}^{II}E_2^{pq} = \mathrm{Ext}_R^p(H_q(P), M)$$

finishing the proof. □

From this we get a direct proof of

Theorem 3.4 (Universal Coefficient Theorem for Cohomology) *Let (P, δ) be a chain complex of projective R-modules such that each $\delta(P_n)$ is also projective. Then for every n and every R-module M, there is an exact sequence*

$$0 \to \mathrm{Ext}_R^1(H_{n-1}(P), M) \to H^n({}_R\mathrm{Hom}(P, M)) \to {}_R\mathrm{Hom}(H_n(P), M) \to 0.$$

Proof. Since $\delta(P_n)$ is projective, the exact sequence

$$0 \to Z_n \to P_n \xrightarrow{\delta} \delta(P_n) \to 0$$

splits, so $P_n \cong Z_n \oplus \delta(P_n)$, showing that Z_n is projective as well. This implies that

$$0 \to \delta(P_{q+1}) \to Z_q \to H_q(P) \to 0$$

is a projective resolution of $H_q(P)$. The 2-stage filtration of the spectral sequence of Proposition 3.3 thus has only nonzero columns for $p = 0$ and 1 and looks like

$$\begin{array}{cc|cc|cc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & {}_R\mathrm{Hom}(H_q(P), M) & \mathrm{Ext}_R^1(H_q(P), M) & 0 & 0 \\
0 & 0 & {}_R\mathrm{Hom}(H_{q-1}(P), M) & \mathrm{Ext}_R^1(H_{q-1}(P), M) & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

The short exact sequence of Lemma 1.8 looks like

$$0 \to \mathrm{Ext}_R^1(H_{n-1}(P), M) \to H^n({}_R\mathrm{Hom}(P, M)) \to {}_R\mathrm{Hom}(H_n(P), M))) \to 0.$$

□

As another example, let us prove following fundamental result in homological algebra by using spectral sequences

Proposition 3.5 *Let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of cochain complexes in an abelian category $\mathcal{A}$. There exists a long exact sequence of cohomology groups

$$\cdots \to H^n(A) \to H^n(B) \to H^n(C) \to H^{n+1}(A) \to \cdots$$

Proof. For the filtration by rows, the first sheet ${}^{II}E_1^{**}$ is zero, so the spectral sequence $\{{}^{I}E_r^{pq}\}$ converges to zero (the filtration is bounded). For the filtration by columns, the first sheet ${}^{I}E_1^{**}$ has the form

$$
\begin{array}{cc|ccccc|cc}
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & H^n(A) & \xrightarrow{f^n} & H^n(B) & \xrightarrow{g^n} & H^n(C) & 0 & 0 \\
0 & 0 & H^{n-1}(A) & \xrightarrow{f^{n-1}} & H^{n-1}(B) & \xrightarrow{g^{n-1}} & H^{n-1}(C) & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots
\end{array}
$$

Then the second sheet ${}^{I}E_2^{**}$ is of the form

$$
\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & ?? \arrow[drr] & & & & \\
0 \arrow[drr] & 0 \arrow[drr] & ?? \arrow[drr] & ? \arrow[drr] & ?? \arrow[drr] & & \\
0 & 0 & ?? \arrow[drr] & ? \arrow[drr] & ?? \arrow[drr] & 0 & 0 \\
& & & & ?? & 0 & 0
\end{tikzcd}
$$

All maps from and to the single question marks are to and from 0-entries, so they stabilize. Because the spectral sequence converges to zero, these questions marks itself have to be zero. This show exactness of

$$H^n(A) \xrightarrow{f^n} H^n(B) \xrightarrow{g^n} H^n(C)$$

After the third sheet, the double-question-mark terms will stabilize as well. Again, because of the convergence to zero, the arrows in ${}^{I}E_2^{**}$ between the double-question-mark terms have to be isomorphisms. Denote them

$$\mathrm{Coker}(g^{n-1}) \xrightarrow{\varphi} \mathrm{Ker}(f^n)$$

This allows us to define a map

$$H^{n-1}(C) \xrightarrow{\partial} H^n(A),\ c \mapsto \varphi([c])$$

We conclude because

$$\mathrm{Ker}(\partial) = \{c | [c] = 0 \text{ in } \mathrm{Coker}(g^{n-1})\} = \mathrm{Im}(g^{n-1})$$

and

$$\mathrm{Im}\partial = \mathrm{Im}\varphi = \mathrm{Ker}(f^n)$$

□

4 Grothendieck spectral sequences

Define for a chain complex (C, δ), the complexes

$$Z'_p = \mathrm{Coker}(C_p \xrightarrow{\delta} C_{p-1}) \quad \text{and } B'_p = \mathrm{Coim}(C_p \xrightarrow{\delta} C_{p-1}).$$

Definition 4.1 Let $\mathcal{A}$ be an abelian category with enough projectives. A left projective resolution or a left Cartain-Eilenberg resolution (CE resolution) P_{**} of a complex A_* is an upper half-plane double complex ($P_{pq} = 0$ if $q < 0$) with augmentation $\varepsilon : P_{*0} \to A_*$ and for which the following left complexes

$$
\begin{array}{ll}
(1)_p & P_{p*} \text{ over } A_p \\
(2)_p & Z_p(P,d^h) \text{ over } Z_p(A) \\
(3)_p & Z'_p(P,d^h) \text{ over } Z'_p(A) \\
(4)_p & B_p(P,d^h) \text{ over } B_p(A) \\
(5)_p & B'_p(P,d^h) \text{ over } B'_p(A) \\
(6)_p & \mathrm{H}_p(P,d^h) \text{ over } \mathrm{H}_p(A)
\end{array}
$$

are all projective resolutions.

Fortunately there's an equivalent (lesser) condition

Proposition 4.2 *If for all p, (4) and (6) are projective resolutions, then P_{**} is a left projective resolution of A_*.*

For the cohomological framework one first defines right Cartan-Eilenberg resolutions of cochain complexes A in an abelian category $\mathcal{A}$ with enough injectives. These are upper-half plane complexes I^{**} of injective objects of $\mathcal{A}$ together with an augmentation $\varepsilon : A^* \to I^{*0}$ such that the maps on coboundaries and cohomology are injective resolutions of $B^p(A)$ and $H^p(A)$. They permit us to define for a left exact functor $F : \mathcal{A} \to \mathcal{B}$ ($\mathcal{B}$ complete) the right hyper-derived functors $\mathbb{R}^i F$ to be $\mathrm{H}^i \mathrm{Tot}^{\Pi}(F(I))$.
For a cochain complex A in $\mathcal{A}$ the two spectral sequences arising from the upper half-plane double cochain complex $F(I)$ become

$$^{II}E_2^{pq} = (R^p F)(\mathrm{H}^q(A)) \Rightarrow \mathbb{R}^{p+q} F(A)$$

and (if A is bounded below)

$$^{I}E_2^{pq} = \mathrm{H}^p(R^q F(A)) \Rightarrow \mathbb{R}^{p+q} F(A).$$

Example 4.3 Let X be a topological space and $\mathcal{F}^*$ a cochain complex of sheaves on X. The hypercohomology $\mathbb{H}^i(X, \mathcal{F}^*)$ is $\mathbb{R}^i\Gamma(\mathcal{F}^*)$ where Γ is the global sections functor. The hypercohomology spectral sequence is $^{II}E_2^{pq} = H^p(X, H^q(\mathcal{F}^*)) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^*)$.

Consider abelian categories $\mathcal{A}$,$\mathcal{B}$ and $\mathcal{C}$ such that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{C}$ be two left exact functors.

$$
\begin{tikzcd}
\mathcal{A} \arrow[rr, "G"] \arrow[dr, "FG"'] & & \mathcal{B} \arrow[dl, "F"] \\
& \mathcal{C} &
\end{tikzcd}
$$

Theorem 4.4 (Grothendieck Spectral Sequence) *Given the above setup and suppose that G sends injective objects of $\mathcal{A}$ to F-acyclic objects of $\mathcal{B}$ (that is, $R^i F(B) = 0$ for $i \neq 0$). Then there exists a convergent first quadrant cohomology spectral sequence in $\mathcal{C}$ for each $A \in \mathrm{Ob}(\mathcal{A})$:*

$$^{I}E_2^{pq} = (R^p F)(R^q G)(A) \Rightarrow R^{p+q}(FG)(A).$$

Proof. Choose an injective resolution $A \to I^*$ of A in $\mathcal{A}$ and apply G to get a cochain complex $G(I)$ in $\mathcal{B}$. Subsequently, use a (right) Cartan-Eilenberg resolution of $G(I)$ to form the hyper-derived functors $\mathbb{R}^n F(G(I))$. Because $G(I)$ is bounded below, there are two spectral sequences converging to these hyper-derived functors. The first spectral sequence is

$$ {}^{I}E_2^{pq} = H^p((R^q F)(G(I))) \Rightarrow (\mathbb{R}^{p+q} F)(G(I)). $$

Each $G(I^k)$ is F-acyclic, so $(R^q F)(G(I^p)) = 0$ for $q \neq 0$. We see that this spectral sequence collapses at $r = 2$ yielding

$$ (\mathbb{R}^p F)(G(I)) \cong \mathrm{H}^p(FG(I)) = R^p(FG)(A). $$

The second spectral sequence is therefore

$$ {}^{II}E_2^{pq} = (R^p F)(\mathrm{H}^q(G(I))) \Rightarrow R^{p+q}(FG)(A). $$

Since $\mathrm{H}^q(G(I)) = R^q G(A)$, this amounts to the Grothendieck spectral sequence. □

Example 4.5 Let R, S be commutative rings, $f : R \to S$ a ring morphism and B an S-module. The functors

$$ {}_R\mathcal{M} \xrightarrow{{}_R\mathrm{Hom}(S,-)} {}_S\mathcal{M} \xrightarrow{{}_S\mathrm{Hom}(B,-)} \underline{Ab} $$

satisfy the hypotheses. Indeed, The first functor is right adjoint to the functor

$$ F : {}_S\mathcal{M} \to {}_R\mathcal{M} $$

which restricts scalars through f. Because kernels and images aren't effected by the R- or S-linearity, this functor is exact, so ${}_R\mathrm{Hom}(S,-)$ is left exact and preserves injectives (see Weibel proposition 2.3.10). The second functor is right adjoint to

$$ G : \underline{Ab} \to {}_S\mathcal{M}; G(N) = B \otimes_{\mathbb{Z}} N. $$

There is a natural isomorphism $FG(A) = {}_S\mathrm{Hom}(B, {}_R\mathrm{Hom}(S,-)) \cong {}_R\mathrm{Hom}(B,-)$ and since both functors are left exact their right derived functors correspond. The Grothendieck spectral sequence thus takes the form

$$ \mathrm{Ext}_S^p(B, \mathrm{Ext}_R^q(S,A)) \Rightarrow \mathrm{Ext}_R^{p+q}(B,A). $$

References

[1] S. Mac Lane, *Homology*, Die Grundlehren der Mathematischen Wissenschaften **114**, Springer-Verlag, Berlin, 1963.

[2] C. Weibel, *An introduction to homological algebra*, Cambridge University Press, Cambridge, 1994.